PRESERVICE ELEMENTARY TEACHERS' MATHEMATICAL CONTENT

KNOWLEDGE OF PREREQUISITE ALGEBRA CONCEPTS

by

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APPROVAL

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the Division of Graduate Education.

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ABSTRACT

Research illustrating that student achievement is affected by teachers' knowledge advocates for K-8 teachers to be knowledgeable regarding prerequisite algebra concepts: (1) numbers (numerical operations), (2) ratios/proportions, (3) the order of operations, (4) equality, (5) patterning, (6) algebraic symbolism (including letter usage), (7) algebraic equations, (8) functions, and (9) graphing. The theoretical framework for the knowledge for teaching mathematics built for this study suggests that the mathematical content knowledge needed for teaching consists of specialized content knowledge in addition to common content knowledge. Specialized mathematical content knowledge extends beyond solving mathematical problems to encompass how and why mathematical procedures work and an awareness of structuring and representing mathematical content for learners.

The effects of an undergraduate mathematics content course for elementary education students on preservice teachers' common and specialized content knowledge of prerequisite algebra concepts was investigated, using a pre-experimental one-group pretest-posttest design. A quantitative, 51-item, multiple-choice instrument, developed specifically to measure both types of content knowledge with respect to prerequisite algebra concepts, was constructed from the Learning Mathematics for Teaching Project's Content Knowledge for Teaching Mathematics Measures question bank. This instrument was administered to all students enrolled in Mathematics for Elementary Teachers I ($n = 48$), at Montana State University, during the fall semester of 2006.

Matched pairs t-tests, comparing pretest and posttest scores within the single sample, show significant gains ($p = .000$) in both common and specialized content knowledge and in all tested aspects of prerequisite algebra knowledge (numbers and equations/functions). Results also suggest a significant correlation ($r = .716$, $p = .000$) between preservice teachers' common and specialized content knowledge. Lastly, a one-parameter linear model was constructed to predict the number of participants to incorrectly answer each item, based on item difficulty. Items missed by notably more or less students than predicted were identified and analyzed. The one item students performed better than expected on addresses common content knowledge regarding a linear graph. The set of troublesome items address both common and specialized content knowledge of reading, writing, and representing functions in a variety of contexts and using ratios to write and solve proportions.

CHAPTER 1

THE PROBLEM

Introduction

Algebra

Ever since algebra became a college entrance requirement at Harvard University in 1820, it has had the ability to establish the educational opportunities available to college-intending students (Moses, 1994; Moses, Kamii, Swap, & Howard, 1989; Picciotto & Wah, 1993). The addition of algebra to increasingly more district and state high school graduation requirements has created a need for all students, no longer just the college-bound, to be algebra proficient (Fey, 1989). Despite the significance of algebra in students' educations and futures, the algebra achievement of U.S. students on the National Assessment of Educational Progress (NAEP) is poor (Chazan & Yerushalmy, 2003).

To address this issue, researchers, teachers, and curriculum experts have worked to identify the prerequisite content areas believed to contribute to students' abilities to succeed in algebra. Specifically, the Southern Regional Education Board (SREB) produced a list of 12 algebra-specific skills, Readiness Indicators, which classify the prior knowledge necessary for success in Algebra I (Bottoms, 2003). The list was developed by mathematics education experts, but not based upon research. Therefore, similarities and differences between relevant research and the Readiness Indicators were investigated

(Welder, 2006). The results of this analysis indicated that nine concepts that are prerequisite to success in a first algebra course are supported by the research:

1. Numbers and numerical operations (Booth, 1984, 1986; Gallardo, 2002; Kieran, 1988; Rotman, 1991; Wu, 2001)

2. Ratios/proportions (Post, Behr, & Lesh, 1988)

3. The order of operations (Kieran, 1979, 1988; Pinchback, 1991)

4. Equality (Behr, Erlwanger, & Nichols, 1976, 1980; Falkner, Levi, & Carpenter, 1999; Herscovics & Kieran, 1980; Kieran, 1981, 1989)

5. Patterning (Watson, 1990)

6. Algebraic symbolism (Behr et al., 1976, 1980; Booth, 1984, 1986; Kieran, 1988, 1992; Küchemann, 1981) including letter usage (Booth, 1984, 1986, 1988; Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Sleeman, 1984; Usiskin, 1988; Watson, 1990)

7. Algebraic equations (Clement, Narode, & Rosnick, 1981; Wollman, 1983)

8. Functions (Brenner et al., 1995)

9. Graphing (Brenner et al., 1995, Chazan & Yerushalmy, 2003; Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990)

The Principles and Standards for School Mathematics published by the National Council of Teachers of Mathematics recommend that these nine concepts, identified above as prerequisite to algebra, are covered within the K-8 mathematics curriculum (NCTM, 2000). Elementary teachers must therefore be prepared to effectively teach these concepts to their students.

Knowledge Needed for Effective Teaching

A teacher's knowledge is one of the biggest influences on classroom atmosphere and on what that teacher's students learn (Fennema & Franke, 1992). In a meta-analysis of 60 education production function studies, variables used to represent teacher quality (such as teacher ability, knowledge, and education level) were found to have positive effects on student achievement (Greenwald, Hedges, & Laine, 1996). Furthermore, the work of Hill, Rowan, and Ball (2005) unveiled that teachers with increased mathematical knowledge for teaching produced significantly larger gains in student achievement, even though the study controlled for many other variables (including student socioeconomic status, student absence rate, teacher credentials, teacher experience, and average length of mathematics lesson). Due to its proven influence, the mathematical knowledge important for the work of teaching has become a significant issue in mathematics education (Stylianides & Ball, 2004).

Over the past two decades, researchers have attempted to identify and categorize the different facets of knowledge obtained and/or required by effective teachers (Ball, 2003, 2006; Hill & Ball, 2004; Hill, Schilling, & Ball, 2004; Rowan, Schilling, Ball, & Miller, 2001; Shulman, 1986). No one disputes that teachers need a thorough understanding of the subject matter they teach; however, focus has been redirected to the additional types of knowledge needed specifically by teachers, as compared to other professionals in their subject area. This work has proposed varying definitions and theoretical frameworks for the knowledge needed for effective teaching. Building off the efforts of Shulman and synthesizing the subsequent work, a theoretical framework

categorizing the knowledge needed for teaching mathematics was developed and guided the remainder of this study. Definitions of key framework terms follow immediately; however, the framework creation will be detailed in Chapter 2 (pp. 34-38).

Theoretical Framework for Knowledge for Teaching Mathematics

I. Mathematical Knowledge for Teaching

    A. Common Content Knowledge (CCK)

    B. Pedagogical Content Knowledge

        i. Specialized Content Knowledge (SCK)

        ii. Knowledge of Students' Conceptual Thinking

        iii. Knowledge of Content and Teaching

        iv. Curricular Knowledge

II. General Pedagogical Knowledge for Teaching

Definition of Framework Terms

- *Knowledge for Teaching Mathematics* – all of the knowledge required to effectively teach mathematics.

- *Mathematical Knowledge for Teaching* – all of the content-specific knowledge required to effectively teach mathematics.

- *Common Content Knowledge (CCK)* – the knowledge of mathematics that allows one to successfully solve mathematical problems.

- *Pedagogical Content Knowledge* – the knowledge of mathematics that allows one to successfully teach mathematical concepts.

- *Specialized Content Knowledge (SCK)* – the knowledge of how and why mathematical procedures and rules work, along with the knowledge of structuring and representing mathematical content. This knowledge also includes the ability to appraise students' ideas and computations and analyze the mathematical validity of unconventional student methods.

- *Knowledge of Students' Conceptual Thinking* – the knowledge of how students analyze and comprehend mathematical ideas. This knowledge entails an understanding of common student conceptions, misconceptions, difficulties, errors, and interests in the field of mathematics.

- *Knowledge of Content and Teaching* – the knowledge of how to select and implement mathematically specific pedagogical strategies (e.g. analogies, illustrations, examples, explanations, demonstrations, manipulatives) to best explain and/or represent mathematical concepts in various instructional situations.

- *Curricular Knowledge* – the knowledge of how the mathematical concepts students are learning relate to the various topics being concurrently addressed in other courses, as well as a holistic awareness of the pertinent mathematics curriculum.

- *General Pedagogical Knowledge for Teaching* – the knowledge of universal educational strategies and principals required to be an effective teacher, including general learning theories and interpersonal skills.

Development of Elementary Teachers' Mathematical Knowledge for Teaching

"Improving the mathematics learning of every child depends on making central the learning opportunities of our teachers," (Ball 2003, p. 9). According to the theoretical framework for knowledge for teaching mathematics, elementary teachers require opportunities to develop two distinct types of mathematical content knowledge concerning prerequisite algebra concepts: common content knowledge and specialized content knowledge.

In the preparation of elementary teachers, mathematical content knowledge is generally addressed throughout one or two required undergraduate mathematics content courses. These courses, specifically designed for preservice elementary teachers, tend to focus on the enhancement of common content knowledge. However, numerous scholars argue that teacher educators have both the responsibility and capability to improve preservice teachers' pedagogical content knowledge within collegiate course settings (Battista, 1994; Stacey et al., 2001; Chen & Ennis, 1995; Manouchehri, 1996; Miller, 1999; Davis & McGowen, 2001). Lee, Meadows, and Lee (2003) claim that in order for preservice teachers to be prepared to teach quality mathematics in their prospective classrooms, teacher educators should ensure that preservice teachers have opportunities to develop mathematical knowledge that is specific to the needs of teachers.

Since undergraduate mathematics content courses for elementary education majors are the only required content courses that will address the mathematical content that preservice teachers will teach, it is vital that they address not only common content knowledge of prerequisite algebra concepts, but also the specialized content knowledge

needed for teaching them. Therefore, this study investigated the ability of a collegiate content course for elementary education majors to develop preservice teachers' common and specialized content knowledge of prerequisite algebra skills.

Statement of the Problem

To succeed in algebra, it is vital that students master prerequisite algebra concepts throughout their K-8 mathematics education. Hence, it is necessary for elementary teachers to be knowledgeable regarding this material. To effectively teach these topics to children, elementary teachers' knowledge must surpass the common content knowledge of prerequisite algebra concepts, to include the specialized content knowledge necessary for teaching them. Collegiate mathematics content courses can address and enhance both of these aspects of the mathematical knowledge needed for teaching prerequisite algebra concepts. Therefore, the current study investigated the effects of an undergraduate mathematics content course on preservice elementary teachers' common and specialized content knowledge of prerequisite algebra concepts.

The purpose of this study was to: (1) develop a quantitative instrument viable for successfully analyzing preservice teachers' common and specialized content knowledge of prerequisite algebra concepts, and (2) implement the developed instrument to measure the effects of an undergraduate mathematics content course for elementary education majors on preservice teachers' common and specialized content knowledge of prerequisite algebra concepts.

It is important to note that the knowledge for teaching mathematics entails knowledge of students' conceptual thinking, knowledge of content and teaching, curricular knowledge, and general pedagogical knowledge for teaching, in addition to mathematical content knowledge (common and specialized). These additional categories of knowledge for teaching mathematics are, however, beyond the scope of this study and were not examined.

Research Questions

The current study focused on several research questions, all with respect to an undergraduate first-semester elementary education mathematics content course. The development and implementation of a quantitative instrument capable of measuring teachers' mathematical content knowledge (both common and specialized) of prerequisite algebra constructs addressed the following questions:

1. What effects does this course have on preservice teachers' mathematical content knowledge of prerequisite algebra concepts?

2. What effects does this course have on preservice teachers' mathematical content knowledge of individual prerequisite algebra constructs (number concepts and equation/function concepts)?

3. What effects does this course have on preservice teachers' common content knowledge and specialized content knowledge of prerequisite algebra concepts?

4. What relationship, if any, exists between preservice elementary teachers' common and specialized content knowledge of prerequisite algebra concepts?

5. What patterns, if any, exist among items missed by more or less preservice elementary teachers than predicted on the instrument measuring mathematical content knowledge of prerequisite algebra concepts?

Significance of the Study

The results of this study have potential to aid collegiate mathematics educators in understanding the common and specialized content knowledge of prerequisite algebra concepts that preservice teachers obtain from their mathematics content courses. This understanding can assist those responsible for developing appropriate curricula for preservice teachers' mathematics content courses. Additionally, prerequisite algebra concepts represent only a small part of the body of mathematical knowledge needed by elementary teachers (NCTM, 2000). The methods and results of this investigation may serve as a basis for further work to examine other areas of mathematical knowledge required of elementary teachers.

CHAPTER 2

REVIEW OF THE LITERATURE

Introduction

Three distinct sections organize the following review of literature. The first section, Prerequisite Knowledge for the Learning of Algebra, includes research that addresses knowledge considered prerequisite for success in algebra, as well as misconceptions of algebra students. This literature was reviewed in an effort to identify concepts students need to be proficient in, prior to entering and being successful in their first algebra course. Nine prerequisite algebra topics were justifiably identified. The second section, Knowledge Needed for Teaching Mathematics, addresses the multifaceted mathematical knowledge that teachers need to effectively teach mathematics. This discussion includes a theoretical framework for the knowledge needed for teaching mathematics, along with justification for its development. The third and final section, Development and Measurement of Specialized Content Knowledge, provides rationalization for addressing specialized content knowledge (in addition to common content knowledge) in collegiate teacher education programs. In addition, appropriate methods for testing such knowledge, including the basis for the quantitative measure that will be designed and utilized in this study, are discussed. Brief summaries are provided at the end of each section, in addition to an overall summary located at the end of the chapter.

Prerequisite Knowledge for the Learning of Algebra

Significance of Algebra

Algebra anchored its existence in the secondary school mathematics curriculum after it became a college entrance requirement at Harvard University in 1820 (Rachlin, 1989). Ever since, algebra has had the ability to determine the educational opportunities available to college-intending students (Moses, 1994; Moses et al., 1989; Picciotto & Wah, 1993). Algebra can separate people from further progress in mathematics-related fields of study (Davis, 1995). However, more and more districts and states have added algebra to their high school graduation requirements causing the need for all students, no longer just the college-bound, to be algebra proficient. For the students who do continue their educations past high school, algebra concepts are necessary for studying every branch of mathematics, science, and technology (Fey, 1989).

Despite the significance of algebra in students' educations and futures, the algebra achievement of U.S. students on the National Assessment of Educational Progress (NAEP) is poor (Chazan & Yerushalmy, 2003). In fact, 53.8% of all responses given on Remedial Intermediate Algebra exams by a group of freshman college students were incorrect (Pinchback, 1991). Pinchback categorized an alarming 40.2% of these incorrect responses as resulting from errors caused by lack of prerequisite knowledge. If lack of preparation for algebra courses is causing poor algebra achievement, then it is essential to identify content whose mastery is required for the learning of algebra.

<u>Defining Algebra</u>

According to Booth (1986), the main purpose of algebra is to learn how to represent general relationships and procedures; for through these representations, a wide range of problems can be solved and new relationships can be developed from those known. However, students tend to view algebra as little more than a set of arbitrary manipulative techniques that seem to have little, if any, purpose to them (Booth, 1986). Perhaps the typical algebra curriculum focuses too heavily on simplification and manipulation, rather than the generalized ideas that create the basis of algebra. Interestingly, the content included in high school algebra has changed very little over the past century (Kieran, 1992).

Standard first-year algebra classes generally include: operations with positive and negative numbers; evaluation of expressions; solving of linear equations, linear inequalities and proportions; age, digit, distance, work and mixture word problems; operations with polynomials and powers; factoring of trinomials, monomial factoring, special factors; simplification and operations with rational expressions; graphs and properties of graphs of lines; linear systems with two equations in two variables; simplification and operations with square roots; and solving quadratic equations by factoring and completing the square (Usiskin, 1987). More concisely, high school algebra can be outlined in five major themes: (a) variables and simplification of algebraic expressions, (b) generalization, (c) structure, (d) word problems, and (e) equations (Linchevski, 1995).

<u>Readiness Indicators</u>

There has been debate regarding the exact identification of concepts needed by a student prior to entering his or her first Algebra course. In one of the most recent papers addressing the issue of identifying prerequisite knowledge for the learning of algebra, the Southern Regional Education Board (SREB) created a panel of classroom teachers and curriculum experts (from the Educational Testing Service) to analyze curriculum materials (Bottoms, 2003). Using their professional expertise, members worked cooperatively to identify 17 mathematical concepts, named Readiness Indicators, believed to classify the skills necessary for a student to be successful in learning Algebra I. The first five Readiness Indicators address general processing skills vital to learning all mathematics. The next 12 Readiness Indicators, however, are content-specific to the learning of algebra (Bottoms, 2003) and are therefore most pertinent to the research topic at hand; they are listed below.

1. Read, write, compare, order, and represent in a variety of forms: integers, fractions, decimals, percents, and numbers written in scientific and exponential notation.

2. Compute (add, subtract, multiply, and divide) fluently with integers, fractions, decimals, percents, and numbers written in scientific notation and exponential form, with and without technology.

3. Determine the greatest common factor, least common multiple, and prime factorization of numbers.

4. Write and use ratios, rates and proportions to describe situations and solve problems.

5. Draw with appropriate tools and classify different types of geometric figures using their properties.

6. Measure length with appropriate tools and find perimeter, area, surface area, and volume using appropriate units, techniques, formulas, and levels of accuracy.

7. Understand and use the Pythagorean relationship to solve problems.

8. Gather, organize, display, and interpret data.

9. Determine the number of ways an event can occur and the associated probabilities.

10. Write, simplify, and solve algebraic equations using substitution, the order of operations, the properties of operations, and the properties of equality.

11. Represent, analyze, extend, and generalize a variety of patterns.

12. Understand and represent functions algebraically and graphically.

The SREB developed the list of Readiness Indicators through much deliberation among experts in the field of mathematics education, however they were not based upon results of research (Bottoms, 2003). In an effort to justify this work, Welder (2006) reviewed literature addressing prerequisite knowledge for the learning of algebra, as well as misconceptions of algebra students. Although the latter does not directly identify prerequisite knowledge, research regarding deficiencies and difficulties of algebra students can provide insight into areas where algebra students are unprepared. Therefore,

this type of research is considered relevant to the discussion of prerequisite knowledge. The following review of literature regarding prerequisite algebra concepts is categorized by algebraic content areas as found in the literature.

Vocabulary

Miller and Smith (1994) identified prerequisite vocabulary for the learning of algebra, due to their belief that lack of prerequisite vocabulary contributes to students' failure to retain problem-solving skills learned in previous mathematics courses. By reviewing course textbooks and interviewing mathematics instructors, they created a 60-item list of vocabulary terms deemed prerequisite for Intermediate Algebra and College Algebra students. This list was then narrowed to 30 items, with the assistance of 44 college mathematics professors from 19 different colleges. The selected vocabulary includes geometric terms such as perimeter, area, volume, and radius, as well as more traditional algebraic terms such as factor, linear equation, slope, and y-intercept. Miller and Smith then investigated Intermediate and College Algebra students' vocabulary, by administering a multiple-choice and true-false vocabulary test; students knew an average of 15 of the 30 terms.

Numbers

Other researchers have focused on the value of understanding numbers prior to algebra introduction (Booth, 1984, 1986; Gallardo, 2002; Kieran, 1988; Rotman, 1991; Wu, 2001). According to Watson (1990), a better understanding of number basics would give students a stronger ability to handle algebraic operations and manipulations. What

types of numbers need to be studied prior to learning algebra? The SREB's Readiness Indicator number 1 focuses on students' ability to read, write, compare, order, and represent a variety of numbers, including integers, fractions, decimals, percents, and numbers in scientific notation and exponential form (Bottoms, 2003). Some of these forms have also been mentioned in research addressing prerequisite number knowledge for the learning of algebra.

Gallardo (2002) focused on the fact that the transition from arithmetic to algebra is where students are first presented with problems and equations that have negative numbers as coefficients, constants, and/or solutions. Therefore, she believes that students must have a solid understanding of integers in order to comprehend algebra. Lack of this understanding will affect students' abilities to solve algebraic word problems and equations. However, Gallardo's research showed that 12- and 13-year-old students do not usually understand negative numbers to the fullest possible extent.

Misconceptions of negative numbers were identified in earlier research done by Gallardo and Rojano (1988; cited in Gallardo, 2002) while investigating how 12- and 13-year-old beginning algebra students acquire arithmetic and algebraic language. One major area of difficulty involved the nature of numbers. Specifically, students had troubles conceptualizing and operating with negative numbers in the context of prealgebra and algebra. Therefore, Gallardo (2002) argues that while students are learning the language of algebra, it is imperative that they understand how the numerical domain can be extended from the natural numbers to the integers.

Kieran (1988) also found misunderstandings regarding integers to affect the success of algebra students in grades 8-11. During interviews with Kieran, students who had taken at least one year of algebra made computational equation-solving errors involving the misuse of positive and negative numbers. Furthermore, when these students were required to use division as an inverse operation, they tended to divide the larger number by the smaller, regardless of the division that was actually required within the operation. Therefore, students' errors extended into the division of integers, which implies a lack of understanding of fractions.

An opinion article regarding how to prepare students for algebra further supports the inclusion of fractions as prerequisite knowledge for the learning of algebra. According to Wu (2001), fraction understanding is vital to a student's transformation from computing arithmetic calculations to comprehending algebra. Wu believes that K-12 teachers are not currently teaching fractions at a deep enough level to prepare students for algebra. In fact, she believes that the study of fractions could and should be used as a way of preparing students for studying generality and abstraction in algebra.

Fractions were also stressed when Rotman (1991) chose number knowledge as a prerequisite arithmetic skill for learning algebra. During a research project that mounted evidence against the assumption that arithmetic knowledge is prerequisite for successful algebra learning, Rotman constructed a list of arithmetic skills he considers as prerequisite to algebra. Based on his experiences as a teacher, Rotman argues that algebra students need to understand the structure behind solving applications, the meaning of symbols used in arithmetic, the order of operations, and basic properties of numbers

(especially fractions). Of course, in order to operate with fractions students are required to know basic number theory ideas including least common multiple. Therefore, the necessity of fraction knowledge partially supports Readiness Indicator number 3, which states that students need to be able to determine the greatest common factor, least common multiple, and prime factorization of numbers (Bottoms, 2003).

Proportionality

Fractions commonly appear in beginning algebra in the form of proportions, which provide wonderful examples of naturally occurring linear functions. Because of this, Post et al. (1988) feel that proportionality has the ability to connect common numerical experiences and patterns, with which students are familiar, to more abstract relationships in algebra. Proportions can also be used to introduce students to algebraic representation and variable manipulation in a way that parallels their knowledge of arithmetic.

In fact, proportions are useful in a multitude of algebraic processes, including problem solving, graphing, translating, and using tables, along with other modes of algebraic representation. Due to its vast utility, Post et al. (1988) consider proportionality to be an important contributor to students' development of pre-algebraic understanding. Similarly, Readiness Indicator number 4 focuses on the importance of ratios, rates, and proportions in the study of algebra (Bottoms, 2003).

Proportional reasoning requires a solid understanding of several rational number concepts including order and equivalence, the relationship between a unit and its parts, the meaning and interpretation of ratio, and various division issues (Post et al., 1988).

Therefore, these concepts could be considered, along with proportional reasoning, prerequisite knowledge for the learning of algebra.

<u>Computations</u>

In addition to understanding the properties of numbers, algebra students need to understand the rules behind numerical computations, as stated in Readiness Indicator number 2 (Bottoms, 2003). Computational errors cause many mistakes for algebra students, especially when simplifying algebraic expressions. Booth (1984) claims elementary algebra students' difficulties are caused by confusion surrounding computational ideas, including inverse operations, associativity, commutativity, distributivity, and the order of operations convention. These misconstrued ideas are among basic number rules essential for algebraic manipulation and equation solving (Watson, 1990). The misuse of the order of operations also surfaced within an example of an error made by collegiate algebra students that Pinchback (1991) categorized as result of lack of prerequisite knowledge. Other errors deemed prerequisite occurred while adding expressions with radical terms and within the structure of long division while dividing a polynomial by a binomial (Pinchback, 1991).

Mentioned by Rotman (1991) as a prerequisite arithmetic skill, the order of operations is also included in Readiness Indicator number 10 (Bottoms, 2003). In fact, this convention has been found to be commonly misunderstood among algebra students in junior high, high school, and even college (Kieran, 1979, 1988; Pinchback, 1991). The order of operations relies on bracket usage; however, algebra requires students to have a more flexible understanding of brackets than in arithmetic. Therefore, according to

Linchevski (1995), prealgebra should be used as a time to expand students' conceptions of brackets.

Kieran (1979) investigated reasons accounting for the common misconception of the order of operations and alarmingly concluded that students' issues stem from a much deeper problem than forgetting or not learning the material properly in class. The junior high school students, with which Kieran worked, did not see a need for the rules presented within the order of operations. Kieran argues that students must develop an intuitive need for bracket application within the order of operations, before they can learn the surrounding rules. This could be accomplished by having students work with arithmetic identities, instead of open-ended expressions (Kieran, 1979).

Although teachers see ambiguity in solving an open-ended string of arithmetic operations, such as $2 + 4 \times 5$, students do not. Students tend to solve expressions based on how the items are listed, in a left-to-right fashion, consistent with their cultural tradition of reading and writing English. Therefore, the rules underlying operation order actually contradict students' natural way of thinking. However, Kieran (1979) suggests that if an equation such as $3 \times 5 = 15$ were replaced by $3 \times 3 + 2 = 15$, students would realize that bracket usage is necessary to keep the equation balanced.

<u>Equality</u>

Kieran's (1979) theory assumes that students have a solid understanding of equations and the notion of equality. Readiness Indicator number 10 suggests that students need to be familiar with the properties of equality before entering Algebra I (Bottoms, 2003). However, equality is commonly misunderstood by beginning algebra

students (Falkner et al., 1999; Herscovics & Kieran, 1980; Kieran, 1981, 1989). Beginning algebra students tend to see the equal sign as a procedural marking that tells them *to do something*, or as a symbol that separates a problem from its answer, rather than a symbol of equivalence (Behr et al., 1976, 1980). Even college calculus students have misconceptions about the true meaning of the equal sign (Clement et al., 1981).

Kieran (1981) reviewed research addressing how students interpret the equal sign and uncovered that students, at all levels of education, lack awareness of its equivalence role. Students in high school and college tended to be more accepting of the equal sign's symbolism for equivalence, however they still described the sign in terms of an operator symbol, with an operation on the left side and a result on the right. Carpenter, Levi, and Farnsworth (2000) further support Kieran's (1981) conclusions by noting that elementary students believe the number immediately to the right of an equal sign needs be the answer to the calculation on the left hand side. For example, students filled in the number sentence $8 + 4 = \square + 5$ with 12 or 17.

According to Carpenter et al. (2000), correct interpretation of the equal sign is essential to the learning of algebra, because algebraic reasoning is based on students' ability to fully understand equality and appropriately use the equal sign for expressing generalizations. For example, the ability to manipulate and solve equations requires students to understand that the two sides of an equation are equivalent expressions and that every equation can be replaced by an equivalent equation (Kieran, 1981). However, Steinberg, Sleeman, and Ktorza (1990) showed that eighth- and ninth-grade algebra students have a weak understanding of equivalent equations.

Kieran (1981) believes that in order to construct meaning while learning algebra, the notion of the equal sign needs to be expanded while working with arithmetic equalities prior to the introduction of algebra. If this notion were built from students' arithmetic knowledge, the students could acquire an intuitive understanding of the meaning of an equation and gradually transform their understanding into that required for algebra. Similarly, Booth (1986) notes that in arithmetic the equal sign should not be read as *makes*, as in *1 plus 2 makes 3*, but instead as *1 plus 2 is equivalent to 3*, addressing set cardinality.

Symbolism

Unfortunately, the equal sign is not the only symbol whose use in arithmetic is inconsistent with its meaning in algebra (Kieran 1992; Küchemann, 1981). In arithmetic, both the equal and the plus sign are typically interpreted as actions to be performed (Behr et al., 1976, 1980); however, this is not how they are used in algebra. In arithmetic, the plus sign becomes a signal to students to conjoin two terms together (as in $2 + \frac{1}{2} = 2\frac{1}{2}$). However, in algebra, $2 + x$ is not equal to $2x$ (Booth, 1986). Both beginning and intermediate algebra students have been found to misinterpret the concatenation of numbers and letters ($4a$) as addition ($4 + a$) instead of multiplication ($4 \times a$) (Kieran, 1988).

To avoid this confusion, Booth (1984) argues that the underlying structure of algebra needs to be exposed to students while working with arithmetic, prior to learning algebra. For example, students are trained throughout arithmetic that solutions are presented in the form of a single term ($2 + 5$ is not an acceptable answer). Therefore,

students believe that signs such as $+$ and $-$ cannot be left in an algebraic answer (such as $3 + a$). This leads to the misuse of concatenation ($3a$) to create an answer that is a single term (Booth, 1988). According to Booth (1984), elementary students should be taught to recognize that the total number of items in two sets containing six and nine objects, respectively, can be written as $6 + 9$ (rather than 15). This will allow them to see how $a + b$ represents the total number of items in two sets (containing a and b items) and can be treated as a single object and valid answer in algebra (Watson, 1990).

Symbolism is mentioned in a substantial portion of the research addressing algebraic understanding and misconceptions (Behr et al., 1976, 1980; Booth, 1984, 1986; Kieran, 1992; Küchemann, 1981); however, is not directly addressed within the Readiness Indicators (Bottoms, 2003). Similarly, letter usage is cited in a great deal of algebra research (Booth, 1984, 1986, 1988; Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Sleeman, 1984; Usiskin, 1988; Watson, 1990); yet, the Readiness Indicators do not directly address this issue either (Bottoms, 2003).

<u>Letter Usage</u>

The transition from arithmetic to algebra can be troublesome for students not only due to symbol confusion, but also because it is where students are first introduced to the usage of letters in mathematics. This new algebraic notation causes difficulties for many students (Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Sleeman, 1984). According to Watson (1990), variable introduction should be based upon pattern generalization. Children should first learn how to find and record patterns and write pattern-rules in words. Eventually they will seek more concise ways of writing rules. At

this time, the introduction of variables will make sense and be appreciated by the student. The extension and generalization of patterns are also noted in Readiness Indicator number 11 (Bottoms, 2003).

Research shows that novice algebra students do not understand the meaning of letters and commonly interpret letters as standing for objects or words (Macgregor & Stacey, 1997). Even once students are able to accept that letters are standing for numbers, they have a tendency to associate letters with their positions in the alphabet (Watson, 1990) and do not understand that multiple occurrences of the same letter represent the same number (Kieran, 1988). After these misconceptions are addressed, students still view the letters as representing specific unknown values, as in $3 + x = 8$, rather than for "numbers in general", as in $x + y = y + x$ (Booth, 1986). Küchemann (1978, 1981) found that only a very small percentage of students, ages 13-15, were able to consider a letter as a generalized number. Even fewer were able to interpret a letter as a variable. The majority of the students in Küchemann's studies treated the letters as concrete objects or just ignored them completely.

Macgregor and Stacey (1997) investigated the origins of students' misinterpretations of letter usage in algebra, throughout a series of studies involving approximately 2000 students, ages 11-15, across 24 Australian schools. This research uncovered that new content students were learning in mathematics and other subjects (such as computer programming) was interfering with their comprehension of algebraic notation. For example, students combined numbers and letters in algebra using rules from the Roman numeration system; some followed the conventions behind writing chemical

combinations in chemistry. In fact, Macgregor and Stacey argue that any alternative letter association can produce misconceptions in students' understanding of algebraic notation. Even the use of letters as a numbering schema in textbooks can cause students to relate letters with their numerical positions in the alphabet.

Misconceptions were also found to be a product of misleading teaching materials. When Macgregor and Stacey (1997) tested students across three schools, multiple times throughout a 13-month period, results showed that students in one school had marked difficulty with letter usage in algebra and persistently misinterpreted letters as abbreviated words or labels for objects. However, in the other two schools, only two instances of letters used as abbreviated words were found in the first test and none after that. It was later realized that teaching materials at the latter two schools only used letters to stand for unknown numbers; whereas those of the first school were found to explicitly present letters as abbreviated words (for example $4d$ could mean "four dogs").

This research supports Booth (1984, 1986, 1988), who argues that student difficulties in beginning algebra result from the inconsistent usage of letters in arithmetic and algebra. In arithmetic, letters such as "m" and "c" are used as labels to represent meters and cents, not the number of meters or the number of cents, as they would in algebra (Booth, 1988). Teachers read the equation $a = l \times w$ as *area equals length times width* (Booth, 1986); yet, they are surprised when students claim that the y in $5y + 3$ could stand for yachts or yams, but must represent something that starts with a y (Booth, 1984).

Furthermore, conversions stated 6m = 600cm are read *six meters are equivalent to 600 centimeters*. Students use this knowledge to read algebraic equations such as $6P = S$ as *six professors are equal to one student* (Booth, 1986). Intuitively this implies that there are six times as many professors as there are students. However, algebraically this equation is representing the exact opposite. This convention could cause students to incorrectly translate word sentences into algebraic equations. In the reverse of the task above, namely symbolizing that there are six times as many students as professors, the most common error is writing the equation $6S = P$, known as a reversal (Wollman, 1983). This translation, however, would make sense to the student who reads it as a conversion statement, *six students are equal to one professor*.

Equation Writing

Translational errors have been identified throughout a variety of equation writing tasks (Clement et al., 1981). A study including 150 freshmen engineering students noted student difficulties in writing equations from data tables. In fact 51% of the students were unable to generate an equation that was being modeled by a table of data. Here, Clement et al. noted the aforementioned misconception of equality, in addition to the occurrence of reversal errors.

Since reversal errors are so common, Wollman (1983) investigated the actions college students take after they write an equation that is reversed. According to Wollman, students lack the ability (or thought) to check their answers in a meaningful way; this inability or negligence is a key component of students' performance in algebra. He suggests that students learn to ask themselves questions regarding the equations that they

write. Upon investigation, not one student in Wollman's six studies could remember being taught how to check the meaning of an equation against the meaning of the sentence it was created from. However, once the students were prompted to think about the equations they had written, many were able to produce correct equations or fix their incorrect ones. Perhaps if the practice of answer checking were taught prior to algebra, it would become a natural part of students' algebraic reasoning and help them in translating various data into algebraic form.

With tools like these, teachers could help strengthen students' fluency in writing equations, a key component of Readiness Indicator number 10. In fact, the SREB acknowledges many of the identified areas of difficulty within this one indicator, which states that students need to be able to write, simplify, and solve algebraic equations using substitution, the order of operations, the properties of operations, and the properties of equality (Bottoms, 2003).

Functions

The SREB also claims that in order to be prepared for Algebra I, students need to understand and be able to represent functions algebraically and graphically, in Readiness Indicator number 12 (Bottoms, 2003). Not only is the concept of a functional relation between two variables a central concept in prealgebra courses, according to Brenner et al. (1995) translating and applying mathematical representations of functional relations are two cognitive skills that are essential for success in algebraic reasoning. Yet, functions are notoriously difficult for many students to understand (Brenner et al., 1995).

One specific difficulty found among ninth- and tenth-grade students who had studied general and linear functions involved using vocabulary terms associated with functions: preimage, image, domain, range, and image set (Markovits, Eylon, & Bruckheimer, 1988). Students also struggled with certain types of functions, such as constant functions and functions whose graphical representations are disconnected. A common misconception was that every function is a linear function.

According to Markovits et al. (1988), students of lower ability find it easier to handle situations involving functions that are given within a story versus those that are only presented mathematically. Although, it should be noted that much research has discussed difficulties that students encounter while solving word problems (Booth, 1981; Chaiklin, 1989; Clement, 1982; Kieran, 1992; Stacey & MacGregor, 2000). Additionally, Markovits et al. concluded that students have an easier time handling functions that are given in graphical form versus those in algebraic form. This result implies that the development of graphing capabilities needs to precede the learning of functions. Similarly, Readiness Indicator number 8 claims that, prior to Algebra I, students should be able to gather, organize, display, and interpret data (Bottoms, 2003), that is be fluent with graphs and tables. However, graphing has been specifically identified as a concept that causes problems for algebra students (Brenner et al., 1995; Chazan & Yerushalmy, 2003; Kieran, 1992; Leinhardt et al., 1990).

Geometry

Readiness Indicators numbers 5, 6, and 7 address geometric concepts including the ability to draw and classify geometric figures, measure length, find perimeter, area,

surface area, and volume, and use the Pythagorean relationship (Bottoms, 2003). Although the current review was not exhaustive, no research-based literature specifically identified geometric skills as prerequisite knowledge or cause of misconception in algebra. However, geometric concepts including area and perimeter have appeared in research investigating algebraic understandings (Booth, 1984; Kieran, 1992; Küchemann, 1978, 1981; Miller & Smith, 1994).

In one such study, Booth (1984) used an item from the Concepts in Secondary Mathematics and Science (CSMS) assessment that involved having students find the area of a rectangle. The rectangle had a height of 7 units, while its length was subdivided into two portions, namely 3 and f. Booth's interviews showed that students had a good understanding of area and could describe their method for finding area verbally; but when the dimensions included variables, students were not able to correctly symbolize their methods or answers. Perhaps basic geometry skills could be used as a foundation to help students build a better understanding of algebra. Additional research is needed to support this idea, along with the claims stated within Readiness Indicators number 5, 6, and 7.

<u>Summary of Prerequisite Knowledge for the Learning of Algebra</u>

Researchers, teachers, and curriculum experts have noted content areas believed to contribute to students' abilities to succeed in algebra. Specifically, the Southern Regional Education Board (SREB) produced a list of 12 algebra-specific skills, Readiness Indicators, which classify the prior knowledge necessary for success in Algebra I (Bottoms, 2003). The list was developed by mathematics education experts, but not based upon research. Therefore, similarities and differences between the relevant

research and the Readiness Indicators were investigated (Welder, 2006). Synthesis of the research indicates that prior to learning algebra, students must have an understanding of: (1) numbers (and numerical operations), (2) ratios/proportions, (3) the order of operations, (4) equality, (5) patterning, (6) algebraic symbolism (including letter usage), (7) algebraic equations, (8) functions, and (9) graphing.

<u>Knowledge Needed for Teaching Mathematics</u>

What a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn (Fennema & Franke, 1992). A recent research study showed that teachers who scored higher on a measure of mathematical knowledge for teaching produced larger gains in student achievement (Hill et al., 2005). In fact, students of teachers who scored in the top quartile showed gains equal to an extra two to three weeks of instruction, when compared with students of teachers who score was considered average (Ball, Hill, & Bass, 2005). Teachers' knowledge successfully predicted the size of student gain scores, even though the researchers controlled for things such as student socioeconomic status, student absence rate, teacher credentials, teacher experience, and average length of mathematics lesson (Hill et al., 2005).

For decades, researchers have been trying to identify the knowledge that is needed by teachers. Although many aspects of teachers' knowledge are agreed upon, the mathematical content teachers must know in order to teach has yet to be mapped precisely (Hill et al., 2004). Hence, multiple legitimate competing definitions of mathematical knowledge for teaching are recognized by the research community (Ball et

al., 2005). Researchers agree that content knowledge (or common knowledge of a subject, not specific to teachers) is an essential aspect of the knowledge needed by teachers. For example, Rech, Hartzell, and Stephens (1993) argue that elementary teachers must possess sound mathematical competency in order to effectively teach mathematics; and, Ma (1999) supports this ascertain, arguing that a profound understanding of fundamental mathematics provides the base for successful mathematics teaching. Ball et al. take this argument one step further by not only stating that the quality of mathematics teaching depends on teachers' content knowledge, but that many U.S. teachers lack firm mathematical understanding and skill.

Knowing mathematics for teaching, however, goes well beyond the knowledge that is needed to reliably carry out an mathematical algorithm (Ball et al., 2005). The daily tasks of teachers, interpreting someone else's error, representing ideas in multiple forms, developing alternative explanations, and choosing usable definitions (Ball, 2003), require teachers to know more than common subject knowledge. A teacher needs principled knowledge of algorithms, solutions, mathematical reasoning, and what constitutes adequate proof, in addition to being skilled in error analysis and the usage of mathematical representations (Ball, 2003; Ball et al., 2005). These types of responsibilities require mathematical reasoning in addition to pedagogical thinking.

Knowledge needed for teaching became a focus in mathematics education research in the mid 1980s, when Shulman and his colleagues (Shulman, 1986; Wilson, Shulman, & Richert, 1987) investigated the subject-matter content needed by teachers in their groundbreaking work regarding the knowledge of accomplished teachers. Their

work introduced a new way of thinking about the content knowledge needed for teaching by developing a framework which categorized the knowledge needed for teaching (Shulman, 1986) and claimed that subject content knowledge must be transformed for teaching (Wilson et al., 1987). Specifically, teachers must have substantial pedagogical content knowledge that can be used to identify useful representations (such as analogies, metaphors, examples, pictorial and physical representations, and practices and drills) to effectively communicate the subject content knowledge to the student (Wilson et al., 1987).

The work of Shulman (1986) and Wilson et al. (1987) extended and challenged the commonly held beliefs about how teachers' knowledge might affect their teaching. These new ideas of teachers' knowledge suggested that teachers' effectiveness is influenced by not only the knowledge of content itself, but also by the knowledge of how to teach that content. Ever since, researchers in mathematics education have increasingly focused on teachers' knowledge of mathematics for teaching. In an extensive synthesis of the research, Hill & Ball (2004) found that studies continue to suggest that in the field of mathematics, how teachers hold knowledge may matter more than how much knowledge they hold. In fact, "teaching quality might not relate so much to performance on standard tests of mathematics achievement as it does to whether teachers' knowledge is procedural or conceptual, whether it is connected to big ideas or isolated into small bits, or whether it is compressed or conceptually unpacked" (Hill & Ball, 2004, p. 332).

Researchers assert that this additional knowledge required of teachers (or lack there of) will affect their teaching decisions and ultimately their students' achievements

in mathematics (Ball & Wilson, 1990; Graeber, 1999; Lee et al., 2003; Rine, 1998). Anders and Leinhardt claim that how teachers link their knowledge to their teaching performance, frequently referred to as pedagogical content knowledge, is critically important for children when they are learning mathematics (as cited in Lee et al., 2003). Ball and Wilson declare that teachers who themselves are tied to a procedural knowledge of mathematics are not equipped to represent mathematical ideas to students in ways that will connect their prior and current knowledge and the mathematics they are to learn, a critical dimension of pedagogical content knowledge. Graeber warns that preservice teachers who enter the classroom without valuing student understanding will not be able to assess understanding or use knowledge of students' current understanding to make instructional decisions. Those who fail to provide alternative paths to understanding will leave some students without understanding. Also, those who fail to recognize and analyze alternative algorithms and solutions will declare students' reasoning incorrect when valid or correct when invalid (Graeber, 1999).

A synthesis of literature done by Rine (1998) further suggests that student achievement can increase if teachers learn about students' thinking. This belief is supported by the work of The Wisconsin University's Cognitively Guided Instruction (CGI) Program and the University of California at Los Angeles' Integrating Mathematics Assessment Project (IMA). Both projects have presented evidence that students whose teachers learned about aspects of students' thinking about addition and subtraction word problems and fractions, respectively, increased their achievement in mathematics (Rine, 1998).

Development of a Theoretical Framework

Researchers have proposed several ways of distinguishing between the many facets of knowledge needed by a teacher to teach effectively. A theoretical framework for the knowledge for teaching mathematics was specifically designed for use in this dissertation, built upon the work of Shulman (1986), Rowan et al. (2001), Hill and Ball (2004), Hill et al. (2004), and Ball (2006).

Shulman (1986) introduced a new way of thinking about the content knowledge needed for teaching in his pioneering work that popularized the concept of pedagogical content knowledge. First, Shulman categorized the knowledge needed for teaching into two domains, namely content knowledge for teaching, "the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9), and pedagogical knowledge, "the knowledge of generic principles of classroom organization and management and the like" (Shulman, 1986, p. 14). Shulman then broke content knowledge for teaching down into three subcategories: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge includes both facts and concepts in a discipline, in addition to why these facts and concepts are true, and an understanding of how knowledge is generated and structured within the subject. "The teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied" (Shulman, 1986, p. 9).

Pedagogical content knowledge, on the other hand, is "the particular form of content knowledge that embodies the aspects of content most germane to its teachability," which "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p. 9). This knowledge includes the ways of representing and formulating the subject that make it comprehensible to others, "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (Shulman, 1986, p. 9). Pedagogical content knowledge also includes an understanding of the conceptions and preconceptions of students, in addition to the factors affecting the difficulty of learning specific topics of study.

Lastly, curricular knowledge is the knowledge of the curriculum which "is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p. 10). This includes knowledge of "alternative curriculum materials for a given subject or topic within a grade", "the curriculum materials under study by his or her students in other subjects they are studying at the same time", and "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school" (Shulman, 1986, p. 10).

According to Rowan et al. (2001), in Shulman's view, pedagogical content knowledge is the form of *practical* knowledge that is used by teachers to guide their actions in highly contextualized classroom settings" (Rowan et al., 2001, p. 2). The multiple aspects of this knowledge were further illuminated when these researchers divided it into three subcategories: (1) content knowledge, (2) knowledge of students' thinking, and (3) knowledge of pedagogical strategies. Content knowledge is the "knowledge of the central concepts, principles, and relationships in a curricular domain, as well as the knowledge of alternative ways these can be represented in instructional situations" (Rowan et al., 2001, p. 5). Knowledge of students' thinking refers to the "knowledge of likely conceptions, misconceptions, and difficulties that students at various grade levels encounter when learning various fine-grained curricular topics" (Rowan et al., 2001, p. 5). Lastly, knowledge of pedagogical strategies is "knowledge of the specific teaching strategies that can be used to address students' learning needs in particular classroom circumstances" (Rowan et al., 2001, p. 3).

Hill and Ball (2004) built off the work of several other researchers (including Bass, Blume, Lamon, Leinhardt, Ma, Simon, Smith, and Thompson, as cited in Hill & Ball, 2004) who studied teacher knowledge in particular topic areas such as fractions, multiplication, division, and rate. These studies helped to clarify what knowing mathematics for teaching requires and ultimately led to the defining of what Hill and Ball term specialized and common content knowledge. In the field of mathematics, common knowledge of content is defined as the basic procedural and conceptual knowledge of solving mathematical problems. "This common knowledge is not unique to teaching;

bankers, candy sellers, nurses, and other nonteachers are likely to hold such knowledge" (Hill & Ball, 2004, p. 333). Specialized knowledge of content, on the other hand, is "unique to individuals engaged in teaching children mathematics" (Hill & Ball, 2004, p. 333). This knowledge includes an ability to explain why procedures work and what they essentially mean, in addition to appraising the methods students use when solving computational problems and determining whether such methods would be generalizable to other problems (Hill & Ball, 2004).

One question that derives immediately from these definitions is whether relationships exist between common mathematical knowledge and specialized mathematical knowledge. Can specialized knowledge for teaching mathematics exist independently from common mathematical knowledge? Analyses of data from large early pilots of surveys with teachers (Hill & Ball, 2004) suggest that it may. Ball et al. (2005) found that the results for the questions representing specialized knowledge of mathematics were statistically separable from results on the common content knowledge items. They believe that these results suggest that there is a place in professional preparation for concentrating on teachers' specialized knowledge.

According to Hill et al. (2004), common and specialized mathematical knowledge comprise one domain of teachers' mathematical knowledge for teaching, namely content knowledge for teaching. The entire framework enclosing the remaining knowledge domains is still under development (Ball, 2006). The most recent version, although unpublished, is built from Shulman's original category scheme (1985, as cited in Ball, 2006). The mathematical knowledge for teaching is divided into two categories: subject

matter knowledge and pedagogical content knowledge. Subject matter knowledge is subcategorized into common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon (knowledge of students' future mathematics curriculum). Pedagogical content knowledge is broken into knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Knowledge of content and students refers to the "knowledge of students and their ways of thinking about mathematics–typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill et al., 2004, p. 17). Knowledge of content and teaching includes appropriate curriculum design and choice of representations, materials, and explanations for students. Lastly, knowledge of curriculum is defined using Shulman's (1986) explanation regarding the curriculum materials in other subjects that a teachers' students are studying at the same time.

The work cited above has been synthesized into the following theoretical framework for knowledge for teaching mathematics, which will serve as a guide for the current study. For detailed definitions of the terms utilized in this framework, please see the definition of terms section in Chapter 1 (pp. 4-5).

Theoretical Framework for Knowledge for Teaching Mathematics

I. Mathematical Knowledge for Teaching

    A. Common Content Knowledge (CCK)

    B. Pedagogical Content Knowledge

        i. Specialized Content Knowledge (SCK)

        ii. Knowledge of Students' Conceptual Thinking

       iii. Knowledge of Content and Teaching

       iv. Curricular Knowledge

II. General Pedagogical Knowledge for Teaching

<u>Summary of Knowledge Needed for Teaching Mathematics</u>

No one disputes that teachers need a thorough understanding of the subject matter they teach. However, over the past two decades, much work has been done to identify and categorize the additional content knowledge needed by effective teachers compared to other professionals in their subject area. A theoretical framework for the knowledge needed for teaching mathematics was developed based on the works of Shulman (1986), Rowan et al. (2001), Hill and Ball (2004), Hill et al. (2004) and Ball (2006). This framework classified the knowledge needed by mathematics teachers into mathematical knowledge for teaching and general pedagogical knowledge for teaching. The mathematical knowledge for teaching was categorized into common content knowledge and pedagogical content knowledge, the latter of which was subdivided into specialized content knowledge, knowledge of students' conceptual thinking, knowledge of content and teaching, and curricular knowledge. The current study will focus on prospective elementary teachers' mathematical content knowledge (both common and specialized) of prerequisite algebra concepts. This knowledge, vital to effective teaching, will be investigated in respect to a collegiate mathematics content course designed for elementary education majors. Although the mathematical knowledge for teaching also entails the knowledge of students' conceptual thinking, knowledge of content and

teaching, and curricular knowledge, these knowledge types are beyond the scope of the current study and will not be addressed within this work.

<u>Development and Measurement of Specialized Content Knowledge</u>

Since research indicates that mathematics teachers need specialized content knowledge of mathematics, it is valuable to investigate where teachers should, or can, develop such knowledge. The arguments and research of numerous scholars provide evidence that it is not only necessary, but in fact possible, for teacher educators to advance preservice teachers' pedagogical content knowledge within collegiate course settings (Battista, 1994; Stacey et al., 2001; Chen & Ennis, 1995; Manouchehri, 1996; Miller, 1999; Davis & McGowen, 2001).

Battista (1994) argues that teacher education institutions need to offer numerous mathematics courses for teachers that treat mathematics as sense making, rather than rule following. Teachers must be taught mathematics properly before they can be expected to teach it properly; and, universities must take the lead in making changes in the way that mathematics is taught. However, Battista warns that simply taking more college-level mathematics courses will not adequately prepare students to teach elementary mathematics. Most university mathematics courses merely reinforce the view of mathematics as a set of memorized procedures; hence, taking more of them will not benefit preservice elementary teachers in the area of specialized content knowledge.

Given the thin pedagogical content knowledge of many preservice teachers, Stacey et al. (2001) recommend that teacher education programs give more attention to

developing this type of knowledge. Specifically, teacher education needs to emphasize pedagogical content knowledge that includes a thorough understanding of common difficulties. Chen and Ennis (1995) support these recommendations by showing that the enhancement of pedagogical content knowledge can help connect the subject content knowledge with the curriculum delivered in classrooms. Furthermore, Lee et al. (2003) claim that in order for preservice teachers to be prepared to teach quality mathematics in their prospective classrooms, teacher educators should ensure that preservice teachers have opportunities to develop mathematical knowledge that is specific to the needs of teachers.

Miller's (1999) research regarding mathematics content courses for preservice teachers implied that mathematical topics covered in these courses should address all three types of Shulman's (1986) teachers' content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Manouchehri (1996) argues that prospective teachers must be given the opportunity in their university course work to strengthen both their content and pedagogical content knowledge. Teacher education programs must provide time to encourage the kind of practice and reflection necessary for the development of prospective teachers' professional knowledge base. One specific area of knowledge that requires attention is the development of representations and representational contexts that will enable teachers to draw connections between concepts and applications, and algorithms and procedures (Manouchehri, 1996).

Hill and Ball (2004) were able to show that elementary teachers advanced their mathematical knowledge for teaching as result of professional development. This work

suggests that policymakers, mathematics educators, and others can successfully design programs that improve teachers' mathematical knowledge for teaching. Unfortunately, "few mathematics courses offer opportunities to learn mathematics in ways that would produce such knowledge" (Ball, 2003, p. 8). University courses required of preservice elementary teachers often do not have the time or concentration needed to develop the mathematical knowledge that is essential for elementary teachers (Battista, 1994).

Practical experience as a teacher was once believed by scholars to be the best way for a person to acquire pedagogical content knowledge. In fact, collegiate teacher education was thought to be incapable of making significant contributions to what teachers need to know or be able to do (Ball & Wilson, 1990). Fortunately, these viewpoints have been challenged and contradicted (Ball & Wilson, 1990; Davis & McGowen, 2001; Miller, 1999; Strawhecker, 2004). Strawhecker (2004) identified significant gains in content knowledge of 96 preservice teachers enrolled in mathematics methods and/or mathematics content courses, at a small Midwestern university. Miller found that a mathematics content course that emphasized all three types of Shulman's (1986) teachers' content knowledge (subject matter content knowledge, pedagogical content knowledge, and curricular knowledge) was effective in increasing all three types of knowledge. Further support of the potential power of content courses is offered by the work of Davis and McGowen, who illustrated that the mathematical understanding of a preservice elementary teacher significantly improved during a mathematics content course. Therefore, research suggests that it is possible to develop specialized content knowledge of preservice elementary teachers within collegiate content course settings.

Studies Utilizing Qualitative Methods

Over the past two decades, researchers have done great amounts of work regarding the knowledge required and/or possessed by successful teachers. Until recently, this type of research most commonly used qualitative methods, hence restricting studies to small sample sizes. Most of these studies relied upon interviews, either alone or in combination with one or more other methods, such as observations and questionnaires.

Ball (1988) examined the knowledge and beliefs of 19 prospective elementary and secondary teachers (10 elementary education majors and 9 mathematics majors or minors preparing to teach high school). These students were interviewed when they were about to enter their first education course, in an attempt to develop a theoretical framework for assessing what teacher candidates bring to their formal preparation to teach mathematics. Although many of the prospective teachers could correctly answer questions regarding the concept of division, several could not, and few were able to explain and connect underlying mathematical principles and meanings. The prospective teachers' knowledge was fragmented, and they tended to search for particular rules (such as "you can't divide by zero") rather than focusing on the fundamental basis of the problems. Even though the mathematics majors showed more mathematical knowledge than the elementary education majors did, they were no more able to provide mathematical explanations and connections.

Chang (1997) also made use of interviews in her evaluation of prospective elementary teachers' mathematical and pedagogical content knowledge. After administering a written test of mathematical content knowledge (adapted from previous

researchers' instruments) to 417 seniors at a teachers' college in Taiwan, she later interviewed thirty of these students. Chang's investigation focused on knowledge of teaching representations, students' strategies, misconceptions, and difficulties, and remediation teaching, as well as the school mathematics curriculum, in the domain of multiplicative structures (multiplication and division, interpretations and relationships of rational numbers, quantitative conceptions, as well as proportionality and linearity). Chang's (1997) results showed that the level of pedagogical understanding of the prospective elementary teachers was unacceptably low (35% correct); and even though the mean score on the test of mathematical content knowledge was better (80% correct), it was not considered entirely satisfactory. Chang concluded that the prospective teachers were not able to represent their teaching methods with a wide variety of models, applied incorrect mathematical knowledge to solve problems, were not willing to prove their formulas, and provided explanations that relied on procedural approaches rather than conceptual understanding.

Several researchers combined the use of interviews with observations in their work to investigate teacher knowledge. In one such study, Foss & Kleinsasser (1996) examined how preservice elementary teachers' beliefs, conceptions, practices, and views of mathematical and pedagogical content knowledge change during their enrollment in a mathematics methods course. Their results revealed symbiotic relationships between teachers' views of content knowledge and instructional procedures that remained problematic.

In another, Swenson (1998) investigated four middle school teachers' subject matter and pedagogical content knowledge of probability and the relationship between these knowledge types and the teaching of probability. The case study began with pre-observation interviews, to assess the probability subject matter knowledge of the teachers. Probability lessons were then observed and videotape-recorded. Then, follow-up interviews explored teacher knowledge and its relationship to teaching practice. Swenson's work showed that the teachers generally (a) lacked an explicit and connected knowledge of probability content, (b) held traditional views about mathematics and the learning and teaching of mathematics, (c) lacked an understanding of the "big ideas" to be emphasized in probability instruction, (d) lacked knowledge of students' possible conceptions and misconceptions, and (e) lacked the knowledge and skills needed to orchestrate discourse in ways that promoted students' higher level learning.

Furthermore, Lomax (1999) inductively analyzed interviews and observations throughout his case study, which indicated that mentors play a critical role in first-year teachers' pedagogical content knowledge growth. The first-year teacher showed pedagogical content knowledge development in her ability to incorporate data from multiple sources to evaluate student knowledge. Growth was noted in her ability to use her pedagogical content knowledge to revise instruction due to changing conditions (in order to tailor instruction to individual student needs) and to contextualize learning situations (so that students could understand the relationships between similar content).

Other researchers have incorporated document analyses into their investigations of teacher knowledge, including those of two aforementioned studies, Miller (1999) and

Davis and McGowen (2001). Miller's conclusion that content courses can be effective in increasing all three types of Shulman's (1986) knowledge for teaching was result of data collected through the written documents of 28 preservice teachers taking their first semester of a mathematics content course. Analyzed documents included initial activities and problem sets, journals, portfolios, task-based assignments, examinations, and course evaluations. Additional data were produced from interviews that Miller conducted with six of the students. Davis & McGowen examined the writings of a student enrolled in a 16-week mathematics content course for preservice elementary teachers. Their evaluations showed that the student began as many preservice elementary teachers do, expecting to apply formulas and get correct answers as in other mathematics classes. However, she made a significant change in her understanding of mathematics as she progressed through the course, leaving with a different, more relational, view of mathematics.

In an additional study, Ward, Anhalt, and Vinson (2003) investigated the development of mathematical representations and pedagogical content knowledge of prospective elementary teachers, by reviewing teachers' written documents as they planned for mathematics instruction. The 31 participating teachers had all completed a prerequisite mathematics content course and were enrolled in a semester-long elementary mathematics methods course at a large Southwestern university. Three times throughout the course, Ward et al. analyzed and coded lesson plans that were submitted as course assignments (not enacted in actual classrooms), based on the planned use(s) of mathematical representations. Over the course of the semester, these preservice teachers

showed an increased ability to make mathematical knowledge usable, by using more representations (the mean number of representations per lesson plan increased from seven to twelve) and moving fluently between the representations in their lessons. These findings led the researchers to believe that the learning opportunities provided in the methods course may have resulted in increased pedagogical content knowledge of the preservice teachers (Ward et al., 2003).

The use of open-ended questionnaires is another qualitative approach that has been utilized by researchers to investigate teachers' knowledge. One such researcher-designed instrument, The Survey on Teaching Mathematics (Rich, Lubinski & Otto, as cited in Fuller, 1996), consists of 12 open-ended questions focusing primarily on whole number operations, fractions, geometry, number sense, and mathematical reasoning. Participants' knowledge and beliefs about mathematics are examined through their responses to questions concerning decisions a teacher would make in specific mathematics classroom situations. Fuller used this instrument to obtain information regarding the pedagogical content knowledge of 26 preservice elementary teachers and 28 experienced kindergarten through sixth-grade teachers. Provided answers were analyzed to characterize and compare preservice and experienced elementary teachers' pedagogical knowledge and pedagogical content knowledge regarding whole number operations, fractions, and geometry. Fuller's results showed that experienced teachers possess a greater conceptual understanding of whole number operations than preservice teachers, but the teachers' knowledge of fractions, in both groups, was primarily procedural. These findings imply that both preservice and experienced teachers (taught in

traditional mathematics classrooms) need to expand their own mathematical knowledge; teachers need opportunities to develop pedagogical content knowledge and to explore, identify, and challenge their assumptions about their role as a mathematics teacher (Fuller, 1996).

To compare the pedagogical content knowledge of mathematics teachers in American and Chinese middle schools, An (2000) administered a questionnaire to 28 mathematics teachers in twelve middle schools in the U.S. and 33 mathematics teachers in 22 schools in China. The questionnaire contained four open-ended questions regarding teachers' pedagogical content knowledge of fractions, ratios, and proportions and eight questions regarding teachers' beliefs of mathematics education. Teachers' responses were coded, grouped, categorized, and compared for data analysis. Additionally, ten teachers, five teachers from each country, were observed and interviewed to clarify their questionnaire responses. An (2000) concluded that the pedagogical content knowledge of middle school mathematics teachers in the U.S. was distinctive from those in China. The U.S. teachers were better able to create various teaching methods, such as connecting to concrete models, cooperative learning, projects, journals, and manipulatives. However, the Chinese teachers, who focused more on conceptual understanding and procedure development, had better knowledge of the students' thinking.

Lastly, Stacey et al. (2001) utilized a Decimal Comparison Test to investigate 553 preservice elementary school teachers' content knowledge and pedagogical content knowledge of decimal numeration. Teachers from four universities in Australia and New Zealand completed the test, where the task was to circle the larger number from pairs of

numbers written in decimal form. Teachers from three universities were then asked to asterisk the comparison items that they believed would be difficult for students and to explain, in writing, on the back of the exam where the difficulty may exist. Stacey et al. concluded that only 80% of the preservice teachers were "experts" in the field of decimal numeration; and that most teachers were aware of the "longer-is-larger" misconceptions of students, but had little awareness of the "shorter-is-larger" misconceptions. In fact, some preservice teachers that acknowledged the "longer-is-larger" misconceptions of students were unknowingly making "shorter-is-larger" errors themselves.

Studies Utilizing Quantitative Methods

Few, if any, of the methods used in the aforementioned teacher knowledge research would be appropriate for use on a large scale. Studies involving a sizeable number of teachers demand the integration of quantitative instruments into methods of measuring teachers' knowledge. Extensive work regarding the mathematical knowledge needed for teaching, conducted by the Learning Mathematics for Teaching (LMT) research group (supported by the National Science Foundation under Grant No. 0335411) at the University of Michigan (LMT, 2006), has begun to address this issue.

Earlier work of Ball and her colleagues (Ball, 1990; Ball & Wilson, 1990) began to incorporate the use of questionnaires into other qualitative methods, interviews and observations, to investigate teacher knowledge. This longitudinal study drew on data from the Teacher Education and Learning to Teach Study at the National Center for Research on Teacher Education at Michigan State University (NCRTE, as cited in Ball, 1990), involving 217 elementary education majors and 35 mathematics majors who

planned to teach high school. This work focused on the mathematical topics of rectangles and squares, perimeter and area, place value, subtraction with regrouping, multiplication, division, fractions, zero and infinity, proportion, variables and solving equations, theory and proof, slope, and graphing.

These preservice teachers were administered a questionnaire upon entrance into, and repeatedly throughout, their formal teacher education (Ball, 1990; Ball & Wilson, 1990). Many of the questions were grounded in scenarios of classroom instruction and involved particular subject matter topics. Although the use of such an instrument was not statistically backed at this time, it previews much of the instrument construction work currently underway by the aforementioned group of LMT researchers (Rowan et al., 2001; Ball & Rowan, 2004; Hill et al., 2004).

A smaller sample of the 252 preservice teachers was followed more closely throughout their preservice program and into their first year of teaching via interviews and observations. Through this work, Ball (1990) examined the preservice teachers' subject matter knowledge by investigating their ideas, feelings, and understandings about mathematics and writing, about the teaching and learning of these subjects, and about students as learners of these subjects. Results showed that the mathematical understanding of elementary and secondary teacher candidates tends to be rule oriented and insufficient. Furthermore, this work led to the comparison of the mathematical understandings and pedagogical content knowledge of beginning teachers entering the field through an alternate route program versus three traditional teacher education programs (Ball & Wilson, 1990). Despite the differences in the teacher education

programs, the novice teachers across the two groups were found to be very similar. Neither the teacher education students nor the teacher trainees in the sample were well prepared to break down the underlying meanings behind mathematical ideas. Neither program had strong influence on novice teachers' ideas about the role of the teacher or practices that would be effective in teaching mathematics. Many of the teachers from both groups were unable to represent basic mathematical content in meaningful ways at the end of their teacher preparation programs.

With the help of projects designed to support teacher knowledge (such as the NSF/DOE Math-Science Partnerships), the LMT researchers at the University of Michigan have made great advancements throughout the last decade in clarifying teachers' knowledge, beginning to track its development, and identifying factors that contribute to its growth (Ball et al., 2005). In 1997, building on earlier work (see Ball & Bass, as cited in Ball et al., 2005) these researchers began examining the work of teaching elementary school mathematics. The *work of teaching* involves "what teachers do in teaching mathematics" and the ways in which "what they do demand mathematical reasoning, insight, understanding, and skill" (Ball, 1993, as cited in Ball et al., 2005, p. 17). The researchers then analyzed this mathematical knowledge required for teaching, in addition to how the knowledge is held and used in the work of teaching.

The LMT research was driven by two questions: (1) "Is there a body of mathematical knowledge for teaching that is specialized for the work that teachers do?" and, (2) "Does it have a demonstrable effect on student achievement?" (Ball et al., 2005, p. 22). From this work they created a practice-based description of what they call

mathematical knowledge for teaching. Mathematical knowledge for teaching is "a kind of professional knowledge of mathematics different from that demanded by other mathematically-intensive occupations (Ball, 2003). As mentioned earlier, this knowledge later came to be categorized into common content knowledge, specialized content knowledge, knowledge at the mathematical horizon, knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum (Ball, 2006; Hill & Ball, 2004; Hill et al., 2004).

To investigate the effect of mathematical knowledge for teaching on student achievement, large data sets were required. The LMT researchers needed to administer numerous items to a large number of teachers to control for the many factors that are likely to contribute to students' learning. Anticipating that samples of a thousand or more teachers might be necessary, these researchers quickly realized that commonly used qualitative methods of measuring teachers' mathematical knowledge (interviews, observations, data analysis, etc.) would not be practical. Therefore, in 1999, LMT began developing unique quantitative measures of teachers' mathematical knowledge for teaching (Ball & Rowan, 2004; Hill et al., 2004; Rowan et al., 2001).

Researchers involved with LMT and the Study of Instructional Improvement (SII), a longitudinal study of high-poverty urban elementary schools engaged in comprehensive school reform efforts, designed a set of survey-based, multiple-choice, teaching problems to measure both the common and specialized content knowledge used in teaching elementary mathematics (Hill & Ball, 2004). Mathematics educators, mathematicians, professional developers, project staff, and former teachers, wrote items

that involved common instructional mathematical tasks, based on the research literature or the writers' experiences teaching and observing elementary mathematics classes (Hill & Ball, 2004; Hill et al., 2005). Questionnaire items presented short, but realistic, scenarios of classroom situations and then asked one or more multiple choice questions regarding these scenarios. Mathematical tasks included representing numbers and operations using materials or stories, providing reasons and explanations for concepts and algorithms, and appraising students' work (Hill & Ball, 2004). Each multiple-choice question contained a "correct" choice and several "incorrect" choices (Rowan et al., 2001).

By the spring of 2001, 138 items had been written to test teachers' knowledge of number concepts (including whole numbers, fractions, and decimals) and operations, patterns, functions, and algebra. Roughly 90 of these items, dispersed among three forms, were piloted during an evaluation of California's Mathematics Professional Development Institutes (Hill & Ball, 2004; Hill et al., 2004); results showed that the measure could both reliably discriminate among teachers and meet basic validity requirements for measuring teachers' mathematics knowledge for teaching. Reliability for the piloted forms averaged in the low .80s, with very few misfitting items. Validation work was also conducted by (a) subjecting a subset of items to cognitive tracing interviews, and (b) comparing items with the NCTM Standards to ensure that the domains specified in these standards were covered (Hill et al., 2005).

This pilot study (Hill & Ball, 2004) tested whether elementary teachers learned mathematics knowledge for teaching during the summer workshop component of

California's K-6 mathematics professional development institutes. Results showed that teachers did learn content knowledge for teaching mathematics from professional development. Additionally, greater performance gains on their measures were found to be related to the length of the professional development and to the curricula that focused on proofs, communication, representations, and solution methods and analyses (Hill & Ball, 2004). Furthermore, factor analysis of the piloted items suggested that teachers' knowledge for teaching elementary mathematics was multidimensional and included knowledge of various mathematical topics (e.g., number and operations, algebra) and domains (e.g., knowledge of content, knowledge of students and content) (Hill et al., 2004).

These measures then linked teachers' mathematical knowledge for teaching to growth in students' mathematical achievement, in the aforementioned study conducted by Hill et al. (2005), using data from a study of schools engaged in instructional improvement initiatives. Researchers had collected survey and student achievement data from students and teachers in 115 elementary schools, across 15 states, during the school years of 2000-2004. Of the 334 first-grade teachers and 365 third-grade teachers included in this study, approximately 90% were fully certified and the teachers averaged just over twelve years of teaching experience apiece. The sample was deliberately constructed, however, to overrepresent high-poverty elementary schools in urban, urban fringe, and suburban areas.

Data from almost 3,000 students were derived from student assessments and parent interviews; measures of student achievement were drawn from CTB/McGraw-

Hill's Terra Nova Complete Battery, the Basic Battery, and the Survey (Hill et al., 2005).

Information about the teachers was gathered from annual questionnaires and written logs

that teachers completed up to 60 times during one academic year. The log was a "self-

report instrument in which teachers recorded the amount of time devoted to mathematics

instruction on a given reporting day, the mathematics content covered on that day, and

the instructional practices used to teach that content" (Hill et al., 2005, p. 381). The

questionnaires contained items about language arts and mathematics teaching,

educational background, involvement in and perceptions of school improvement efforts,

and professional development, in addition to a total of 30 items (taken from the roughly

90 piloted in Hill et al., 2004) designed to measure teachers' mathematical knowledge for

teaching. These items were balanced across content domains (13 number items, 13

operations items, and 4 pre-algebra items) and across specialized (16 items) and common

(14 items) content knowledge.

Summary of Development and Measurement of Specialized Content Knowledge

Discussion and research of numerous scholars provide substantiation of the

necessity and ability of teacher educators to enhance preservice teachers' pedagogical

content knowledge within collegiate course settings (Battista, 1994; Stacey et al., 2001;

Chen & Ennis, 1995; Manouchehri, 1996; Miller, 1999; Davis & McGowen, 2001). In

the field of mathematics, elementary education majors are generally required to take only

one or two semesters of mathematics content courses, specifically designed for

elementary education majors, in addition to a mathematics methods course. Since these

are the only required courses that will address the mathematical content that these

preservice teachers will teach, it is vital that they address not only common but also specialized content knowledge. Therefore, this study investigated the gains in mathematical content knowledge, both common and specialized, produced by an undergraduate mathematics content course for elementary education majors. Although research has shown that methods courses have also been successful in increasing the pedagogical content knowledge of teachers, these courses focus largely on pedagogical strategies and therefore were not included in this study of content knowledge.

Researchers have implemented various methods of examining and testing the pedagogical content knowledge of preservice and inservice teachers (An, 2000; Chang, 1997; Foss & Kleinsasser, 1996; Fuller, 1996; Hill et al., 2005; Miller, 1999; Stacey et al., 2001). Although the majority of these methods are qualitative, recent research involving large quantities of teachers has necessitated the development of appropriate quantitative measures of teachers' knowledge. Since a sample size of 60-70 preservice elementary teachers was anticipated for this study, the aforementioned literature justified the development and usage of a multiple-choice measure of teachers' common and specialized content knowledge constructed from the items written by the LMT research group at the University of Michigan (Ball & Rowan, 2004).

<u>Summary</u>

Based on the reviewed literature, there are nine mathematical concepts that students must be knowledgeable in, prior to entering and being successful in their first algebra course: (1) numbers (and numerical operations), (2) ratios/proportions, (3) the

order of operations, (4) equality, (5) patterning, (6) algebraic symbolism (including letter usage), (7) algebraic equations, (8) functions, and (9) graphing (Welder, 2006). Since NCTM Standards command that the K-8 mathematics curriculum cover these topics (NCTM Principles & Standards, 2000), prerequisite algebra concepts become the responsibility of elementary school teachers. Since research shows that teachers' knowledge affects student achievement (Greenwald et al., 1996; Hill et al., 2005), elementary teachers must have adequate knowledge to effectively teach prerequisite algebra concepts.

Even though a profound understanding of fundamental mathematics is essential to successful mathematics teaching (Ma, 1999), effective teachers must also possess mathematical knowledge that "goes well beyond what is needed to carry out (an) algorithm reliably" (Ball et al., 2005, p. 22). Teachers need specialized mathematical knowledge that is specific to the daily tasks of teachers, including "interpreting someone else's error, representing ideas in multiple forms, developing alternative explanations, and choosing a usable definition" (Ball, 2003, p. 8). A theoretical framework for the knowledge for teaching mathematics (built upon the work of Hill & Ball, 2004; Hill et al., 2004; Shulman, 1986; Rowan et al., 2001) suggests that the mathematical content knowledge needed for teaching consists of not only common content knowledge, but also specialized content knowledge.

Specialized content knowledge has been found to increase as result of collegiate mathematics content courses (Davis & McGowen, 2001; Miller, 1999), field experiences (Strawhecker, 2004), and professional development seminars (Hill & Ball, 2004). This

research suggests that it is appropriate and necessary for specialized content knowledge to be addressed within collegiate courses for preservice teachers. To examine and test the pedagogical content knowledge and specialized content knowledge, researchers have utilized various methods, most of which have been qualitative (An, 2000; Chang, 1997; Foss & Kleinsasser, 1996; Fuller, 1996; Hill et al., 2005; Miller, 1999; Stacey et al., 2001). However, recent research involving large quantities of teachers has motivated the development and piloting of reliable quantitative measures of the mathematical knowledge needed for teaching (Ball & Rowan, 2004; Hill et al., 2004; Rowan et al., 2001).

In conclusion, students must learn prerequisite algebra concepts throughout their K-8 mathematics education, making it necessary for elementary teachers to be knowledgeable regarding this material. To effectively teach these topics to children, elementary teachers' knowledge must surpass the common content knowledge of prerequisite algebra concepts, to include the specialized content knowledge necessary for teaching them. Collegiate mathematics content courses can address and enhance both of these aspects of the mathematical knowledge needed for teaching. Therefore, the current study investigated the effects of an undergraduate mathematics content course on preservice elementary teachers' common and specialized content knowledge of prerequisite algebra concepts. Since a sample size of 60-70 participants was anticipated for this study, the aforementioned LMT quantitative measures were used in the design and implementation of a instrument which tests preservice teachers' common and specialized content knowledge of prerequisite algebra concepts (Ball & Rowan, 2004).

CHAPTER 3

DESIGN AND METHODS

Research Design

The focus of this study was twofold, and included both: (1) the development of a quantitative instrument viable for successfully analyzing teachers' content knowledge (both common and specialized) of prerequisite algebra concepts, and (2) the implementation of the developed instrument to measure the effects of an undergraduate mathematics content course for elementary education majors on preservice teachers' mathematical content knowledge (both common and specialized) of prerequisite algebra concepts (see p. 2). The second portion of this study examined gains in mathematical content knowledge through a pre-experimental one-group pretest-posttest design.

Research Questions

The current study focused on several research questions, all with respect to an undergraduate first-semester elementary education mathematics content course. The development and implementation of a quantitative instrument capable of measuring teachers' mathematical content knowledge (both common and specialized) of prerequisite algebra constructs addressed the following questions:

1. What effects does this course have on preservice teachers' mathematical content knowledge of prerequisite algebra concepts?

2. What effects does this course have on preservice teachers' mathematical content knowledge of individual prerequisite algebra constructs (number concepts and equation/function concepts)?

3. What effects does this course have on preservice teachers' common content knowledge and specialized content knowledge of prerequisite algebra concepts?

4. What relationship, if any, exists between preservice elementary teachers' common and specialized content knowledge of prerequisite algebra concepts?

5. What patterns, if any, exist among items missed by more or less preservice elementary teachers than predicted on the instrument measuring mathematical content knowledge of prerequisite algebra concepts?

Pilot Study

To assist the design of the research methods, a pilot study was conducted to investigate preservice teachers' content knowledge of prerequisite algebra concepts. The pilot sample consisted of all students enrolled in Math 130 ($n = 55$) or Math 131 ($n = 58$) at Montana State University, present in class on April 10, 2006 ($n = 113$). The students were administered two of the Content Knowledge for Teaching Mathematics (CKTM) Measures created through the Learning Mathematics for Teaching (LMT) Project at the University of Michigan (LMT, 2006). CKTM forms addressing number concepts and operations (NCOP) have piloted reliabilities that range from 0.71-0.89; those addressing patterns, functions, and algebra (PFA) range from 0.77-0.87. The measure administered

in the pilot study combined the most recent form developed in each of these content areas (NCOP 2004B & PFA 2006B). All measures were hand-scored using the raw score to z-score conversion charts that were constructed and provided by the LMT researchers based on the performance of the sample in the LMT pilot studies (Hill & Ball, 2004; Hill et al., 2004). Since two z-scores were created for each participant (one for NCOP and one for PFA), a total of 226 z-scores were calculated. The results of this pilot investigation guided the remaining design of the current research study.

<u>Sample</u>

Elementary education majors at Montana State University are required to take one year-long sequence of mathematics content courses: Mathematics for Elementary School Teachers I and II (Math 130 and 131). The first semester of this sequence (Math 130) addresses sets, whole numbers (operations, properties, and computations), number theory, fractions, decimals, ratios, proportions, percents, integers, and sometimes rational and real numbers (Musser, Burger, & Peterson, 2006). The curriculum of Math 131, contrastingly, focuses on geometry, statistics, and probability.

Due to the varied aims of these courses, Math 130 is the only content course that directly addresses any of the nine prerequisite algebra concepts (numbers and numerical operations, ratios/proportions, the order of operations, equality, patterning, algebraic symbolism including letter usage, algebraic equations, functions, and graphing). Although Math 131 does offer students opportunities to practice and enhance prerequisite algebra skills, it is hypothesized that this course does not significantly increase students'

content knowledge in regards to prerequisite algebra concepts. The results of the pilot study support this assumption by showing that students who have taken both Math 130 and Math 131 differ little in terms of their knowledge of prerequisite algebra concepts from students who have only taken Math 130. A one-way ANOVA illustrated that the pilot data did not provide significant evidence ($p = 0.193$, $\alpha = 0.05$) to reject a null hypothesis that the mean of the difference in NCOP content knowledge between Math 130 and Math 131 students is zero.

Similar results showed that although Math 131 students achieved higher scores of PFA content knowledge, this difference was also not significant ($p = 0.067$, $\alpha = 0.05$). It is important to note that the traditional track of these two mathematics content courses involves taking Math 130 during a fall semester and Math 131 in the subsequent spring semester. Therefore, the non-traditional courses (Math 130 courses offered in the spring and Math 131 courses offered in the fall) tend to have a higher percentage of students who are retaking the course. This fact would predict the achievement level of students in a fall Math 130 course to be greater than that of a Math 130 course offered in the spring. Hence, even though there appeared to be a small, but not statistically significant, difference in PFA content knowledge, this result is believed to be an effect of the pilot study taking place during a spring semester.

Math 130 is the best opportunity for preservice teachers to enhance not only their common content knowledge, but also their specialized content knowledge of prerequisite algebra concepts. In fact, due to the variety of collegiate methods courses and experiences working with children afforded to students, this course could be the only

exposure to prerequisite algebra concepts some preservice teachers get before entering the work field. Therefore, it is paramount that the effectiveness of Math 130 in increasing preservice teachers' mathematical content knowledge (both common and specialized) of prerequisite algebra concepts is examined. Thus, the sample of this study was comprised of all students who completed Math 130 at Montana State University, during the fall semester of 2006 ($n = 48$). With only minor variations, these students were mostly female and freshman of traditional age.

Discussion of Math 130 Design and Instruction

Construction

Math 130 is a four-credit semester course which meets for 50-minute time periods, four days a week (Monday, Tuesday, Thursday, and Friday), for approximately 16 weeks. Three sections of the course were offered during the semester of data collection (fall 2006), from 8:00-8:50am, 9:00-9:50am, and 12:00-12:50pm. Course material is examined through a variety of instructional strategies including lecture, class discussion, hands-on activities, group-work and student collaboration, student presentations, writing tasks, quizzes, and exams.

Instructors

During the fall of 2006, graduate teaching assistants independently taught two of the three sections, while the third, as well as course supervision, was handled by and assistant professor of mathematics education. Instructors met one hour a week to design course activities and exams and to align course schedules and goals. Although instructors

wrote individual quizzes, the syllabus, most activities, and all exams given across the three sections were identical.

Course Objectives

As stated on the course syllabus:

1. Solve mathematical problems based on Polya's model and using a variety of strategies.

2. Identify the structure of the whole, integer, rational, and real number systems.

3. Perform mathematical operations in base ten and other bases, use traditional and alternative algorithms, and solve elementary problems in number theory and set theory.

4. Apply technology appropriately in exploring and solving mathematical problems.

5. Model and use an activity-oriented approach to teaching and learning mathematics.

6. Encourage discourse, self-motivation, and independent thinking in learning mathematics.

Course Materials

The Math 130 course curriculum, which is considered to be standard for this type of mathematics content course offered for elementary education majors, sequentially followed Chapters 1-9 of the textbook, *Mathematics for Elementary Teachers: A Contemporary Approach, 7th edition* (Musser et al., 2006).

During the fall of 2006, the Math 130 curriculum excluded Section 4.3 (Algorithms in Other Bases), combined Sections 2.4 (Relations and Functions) and 9.3 (Functions and Their Graphs) and covered them together after Chapter 3 (of Musser et al., 2006), and reversed the order of Sections 9.1 (The Rational Numbers) and 9.2 (The Real Numbers). Additional student activities and supplementary materials used in the course came from multiple sources, including Dolan, Williamson, and Muri (2007), Friel, Rachlin, and Doyle (with Nygard, Pugalee, and Ellis) (2001), Johnston (1998), Lappen,

Fey, Fitzgerald, Friel, and Phillips (1998), Willard (2006), and Williams and Bright (1998). Instructors had no knowledge of the items being tested by this study; therefore, instruction was totally disconnected from the instrument.

<div align="center">Measure</div>

An instrument was developed to test preservice teachers' common and specialized content knowledge of prerequisite algebra concepts. The instrument was constructed from the Content Knowledge for Teaching Mathematics Measures (CKTM measures), created by the Learning Mathematics for Teaching (LMT) Project (Hill et al., 2004). Select items from the CKTM measures question bank (addressing number concepts, number operations, patterns, functions, and algebra) were combined to create a measure addressing each of four constructs (common content knowledge, specialized content knowledge (see pp. 4-5), number concepts, and equation/function concepts (see pp. 71-72)). All items included in these four individual measures were then combined to create the final instrument addressing common and specialized content knowledge of prerequisite algebra concepts.

Item response theory was utilized in the development of this instrument. Item response theory estimates abilities for the individuals that take a test and item parameters based on abilities. The LMT researchers used this theory to create information for each item included among their various pilot studies. Each item is illustrated in terms of an item characteristic curve, which displays the probability that a teacher (based on his/her ability level) will get that item correct (see Figure 1). Ability levels are written as

standard deviation units ranging from -3.0 to +3.0 standard deviations. Item information,

provided by the authors in scalebuilding spreadsheets, was used to construct hypothetical

test information curves for each of the four embedded measures and the overall

developed instrument to project internal reliability.

Figure 1. Example item characteristic curve.
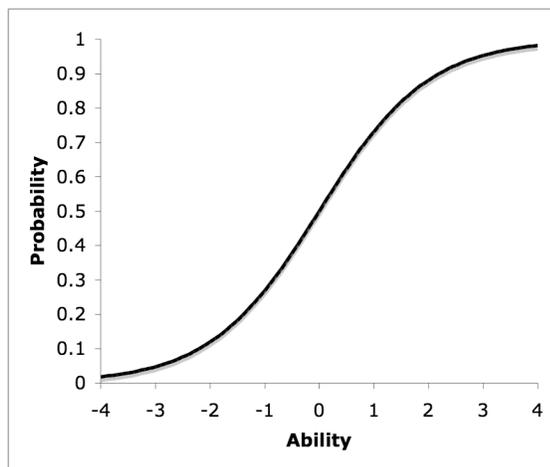
Targeting the Assessment

Each measure needed to be developed to assess teachers well across the range of

ability levels that were projected within the sample. If the instrument was not designed

around the projected ability levels of the teachers in the sample, it could have potentially

lost the power to discriminate between teachers of certain ability levels. For example, if

an exam is to be comprised of five items ($I_1 - I_5$) and given to two teachers (T_1 and T_2),

then it must be designed to discriminate between the ability levels of these two teachers.

The following curves (see Figures 2 and 3) illustrate the distribution of teacher ability

levels, with five potential exam items marked along the teacher ability levels they target

for each figure. Since the items marked in Figure 2 are designed to target teachers of lower abilities than the teachers in the sample, an instrument only comprised of these items would not be capable of discriminating between these two teachers. Both of these teachers would be expected to get all five items correct, producing equivalent test scores. However, if the items were chosen to spread more thoroughly across the range of abilities in which these teachers fall (as in Figure 3), the instrument would have a better potential to discriminate among the two teachers. For the five items marked in Figure 3, teacher two (T_2) would be expected to get three items correct, whereas teacher one (T_1) would be expected to get only two items correct.

Figure 2. Five exam items incapable of discriminating between the two given teachers.
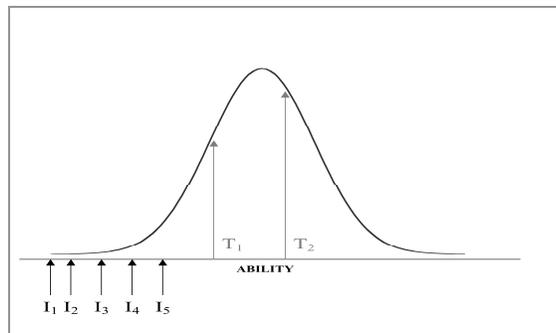
Figure 3. Five exam items capable of discriminating between the two given teachers.
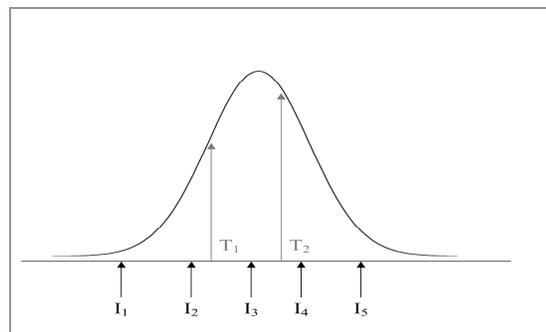
LMT researchers believe that a typical sample of teachers will perform within the range of -2.0 and +2.0 standard deviations of the teachers that they evaluated in their pilot studies. However, all of the CKTM items were piloted with inservice teachers whereas this study's sample consisted of preservice teachers. Therefore, it was necessary to estimate how the average preservice teacher in the sample was expected to perform relative to the average inservice teacher in the LMT pilot studies.

The results of the aforementioned pilot investigation showed that the sampled population of preservice teachers performed approximately 0.25 standard deviations below the pilot sample (see Figure 4 for distribution of the 226 z-scores). Based on this information, the instrument developed was designed to measure preservice teachers ranging in ability between -2.25 and +1.75 standard deviations well, with the ability level of -0.25 standard deviations targeted for best measurement.

Figure 4. Results of the pilot study administration of two CKTM Measures ($n = 113$) displayed in standard deviation units.
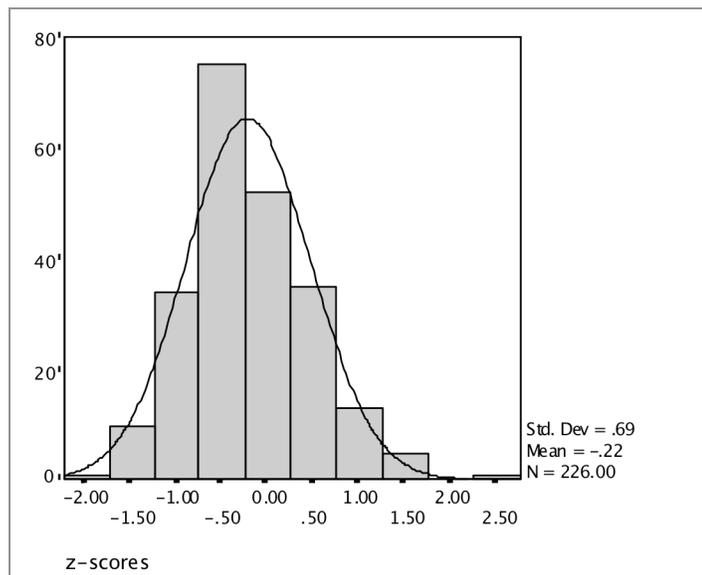
<u>Item Consolidation and Coding</u>

Approximately 375 items from 23 preconstructed elementary and middle schools CKTM forms were gathered together. Items that did not address number concepts, numerical operations, patterns, functions, or algebra, as well as repeat items, were deleted; nearly identical items were combined. All of the remaining unique items were considered for inclusion on the developed instrument.

These items were coded according to the type of knowledge (common versus specialized, see pp. 4-5) addressed in the question and the aforementioned nine prerequisite algebra concepts (see p. 2) that they address. Since items consistently addressed multiple prerequisite algebra concepts simultaneously, areas of overlap were naturally condensed into two prerequisite algebra constructs: (1) number concepts, and (2) equation/function concepts. These constructs, described below, were then expert-checked by a local eighth-grade prealgebra teacher.

Number Concepts are skills related to reading, writing, representing, and computing with numbers in a variety of forms, including integers, fractions, decimals, ratios, and proportions. Since correct usage of the order of operations is vital to numerical computations, this concept is also included in this construct.

Equation/Function Concepts entail a conceptual understanding of variables, in addition to an ability to express generalizations, represent situations algebraically, simplify and solve algebraic representations (including linear equalities and inequalities), use formulas, and understand the relationship between an equation and its graphical representation. These tasks require a proper understanding of algebraic symbolism,

including an expanded interpretation of both the plus and equal signs and letter usage in algebra. Further ideas relate to the functional relationship between two variables. Skills include the ability to determine output of a function, work with functions created by familiar formulas, understand rates of change, differentiate between linear and non-linear functions, and understand the relationship between a function, its graph, and its information presented in tabular form. Since teachers commonly use the analysis and generalization of patterns to introduce students to functional relationships, patterning ideas are also included under this construct.

<u>Item Selection</u>

The developed instrument examines four unique constructs: common content knowledge, specialized content knowledge, number concepts, and equation/function concepts. A total of 51 items were needed to produce an instrument with optimal testing abilities (see discussion on pp. 79-80) across all four constructs. The LMT researchers suggest that participants are given one minute of testing time per item (or two minutes per stem) administered. Therefore, it was possible to administer the instrument to Math 130 students during one 50-minute class meeting.

For each construct, items were selected based on three criteria: (1) the construct addressed by the item, (2) the difficulty of the item, and (3) the slope of the item's characteristic curve at its difficulty level. The characteristic curve of an item shows the probability that a teacher (based on his/her ability level) will get that item correct (see Figure 1). Item difficulty is defined as the point at which the item's characteristic curve crosses the 50% probability level. The item displayed in Figure 1 has a difficulty level of

0.0 (see Figure 5), meaning that teachers who perform around 0.0 standard deviation units from the mean teacher ability would have a 50% chance of getting the item correct.

Figure 5. Item characteristic curve for an example item with difficulty level of 0.0.
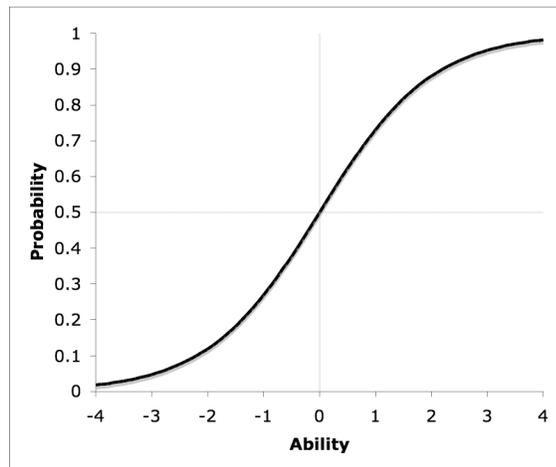
Items were chosen so that the difficulty levels of those items were distributed well across the projected ability range of the sample. Since the ability level of -0.25 standard deviations was targeted for best measurement for this instrument, many items with difficulty levels between -0.75 to +0.25 standard deviations were chosen. However, some items of higher and lower difficulty levels were also included so that the instrument is capable of measuring teachers of higher and lower ability levels.

Lastly, the slope of an item's characteristic curve at any given ability level shows how well the item discriminates among teachers of that ability level. The slope of this curve will always be steepest at the item's difficulty level; therefore, items will always best discriminate among teachers whose ability levels are near the difficulty level of the item. However, not all items discriminate well. The steeper the slope of an item's

characteristic curve at its difficulty level, the better the item discriminates among teachers at ability levels near its difficulty level. For example, the following two characteristic curves, shown in Figure 6, illustrate two items which both have a difficulty level of 1.0. Both items best discriminate among teachers whose ability levels are close to +1.0 standard deviations above the mean, however they do not discriminate equally well among these teachers.

Figure 6. Item characteristic curves of two example items that have equal difficulty levels (1.0) but vary in their ability to discriminate due to the difference in their slopes.
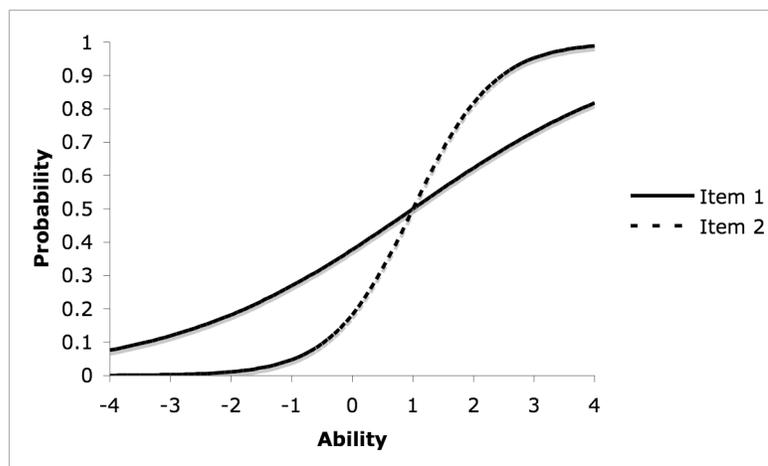
For item 1, a teacher of ability level 0.0 would have a 40% chance of getting the item correct, whereas a teacher of ability level 2.0 would have a 60% chance of getting it correct. These two probabilities vary little considering the teachers' ability levels differ by two standard deviations. Contrastingly, the probability of a teacher of ability level 0.0 getting item 2 correct is less than 20%, whereas the probability for a teacher of ability level 2.0 increases to 80%. Since it is more likely that teachers of these two ability levels

would differ in the correctness of item 2 versus item 1, item 2 would have a stronger chance of discriminating between teachers of these two ability levels.

The slope of the item characteristic curve at a particular ability level is considered the amount of information generated by that item for teachers of that ability level. Therefore, another way to view an item is by examining its item information curve, which illustrates the amount of information generated by the item as a function of teacher ability level (see Figures 7 and 8 for the two corresponding curves of the item illustrated in Figure 1). Since the item information curve shows the slope of the item characteristic curve at any given teacher ability level, it is the derivative of the item characteristic curve.

Figure 7. Example item characteristic curve.
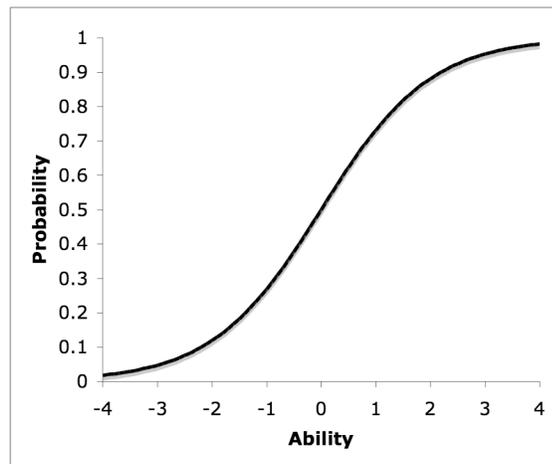
Figure 8. Example item information curve.
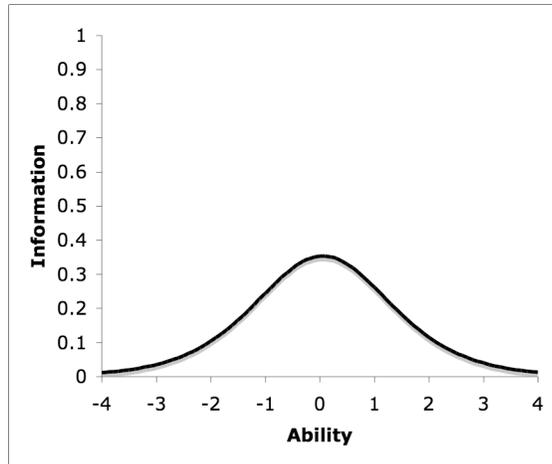
An item's characteristic curve will always have its steepest slope at its difficulty level, inducing the item's information curve to have a maximum value corresponding with the difficulty level of the item. For example, the item displayed in Figures 7 and 8 has a predicted difficulty level of 0.0 standard deviations (see Figure 5), therefore generating the most information for teachers of that ability level. This item will, however, generate very little information about a hypothetical teacher whose ability level is -2.5 standard deviations. Therefore, preference was given to items whose information curves showed the highest production of information (ensuring the selection of items whose characteristic curves have steeper slopes).

Item Analysis

Initial selected items were located among the scalebuilding spreadsheets that the LMT researchers built from the results of their pilot studies. Each spreadsheet is constructed with items listed across the top, ordered by difficulty, and a series of

hypothetical teacher ability levels (ranging from -3.0 to +3.0 standard deviations) listed down the left side. For each item, the projected information generated (or slope of the item characteristic curve) is provided for each ability level in table form (an example set of seven items can be seen in Figure 9).

Starting with the lowest ability level provided (-3.0 standard deviations), the projected test information generated by the set of selected items for teachers at each ability level can be calculated. Since item information curves are additive, the information generated at this ability level for each of the selected items can be added together in a new column of the spreadsheet. For example, the information generated by each of the seven items shown in Figure 9, for a teacher of ability -3.0 standard deviations, have been added together in a new column (see Figure 10). This sum (1.073 units of information) represents the projected total information generated by a measure composed of these seven selected items for a person with an ability level of -3.0 standard deviations. That is, the test information curve for an exam containing these seven items will show a height of 1.073 units at the ability level of -3.0, meaning that the test characteristic curve has a slope of 1.073 at the ability value of -3.0 standard deviations.

The "fill down" feature of Excel easily calculates the projected information generated by the selected items for the remaining ability levels. This newly created column now represents the functional values of the projected test information curve (for an instrument consisting of the selected items), calculated from a two-parameter model using both difficulty and slope of the selected items. The test information curve can then be graphed relative to teacher ability levels, by selecting both the first column, containing

Figure 9. Scalebuilding spreadsheet for seven example items.

| ITEM	A2C	C15	B1B	C2A	C18B	A7	A5A
SLOPE	0.29781	0.8911	0.95936	0.6343	0.88444	0.38305	0.15323
DIFF	-3.26838	-2.53874	-1.69971	-1.16104	-0.85548	1.04519	1.31353
ABILITY LEVEL							
-3.0	0.0637843	0.5090076	0.2543733	0.1236716	0.0831797	0.0264969	0.0125633
-2.9	0.0635251	0.5328087	0.288387	0.1340341	0.0954862	0.0280265	0.0127294
-2.8	0.0631867	0.5518155	0.3250672	0.1448758	0.1094041	0.0296279	0.0128945
-2.7	0.0627707	0.565233	0.3640368	0.1561413	0.1250762	0.0313023	0.0130584
-2.6	0.0622792	0.5724744	0.4047197	0.1677581	0.1426357	0.0330505	0.0132209
-2.5	0.0617147	0.5732141	0.446321	0.1796353	0.1621965	0.0348734	0.0133819
-2.4	0.0610798	0.5674186	0.4878246	0.1916636	0.1838418	0.0367711	0.0135412
-2.3	0.0603777	0.5553493	0.528012	0.2037148	0.2076102	0.0387435	0.0136987
-2.2	0.0596116	0.5375383	0.5655091	0.2156435	0.2334793	0.0407902	0.0138543
-2.1	0.0587852	0.5147392	0.5988615	0.2272883	0.2613477	0.0429099	0.0140077
-2.0	0.057902	0.487862	0.6266365	0.2384757	0.2910164	0.0451011	0.0141589
-1.9	0.0569662	0.4579009	0.6475406	0.2490234	0.3221718	0.0473614	0.0143076
-1.8	0.0559816	0.4258633	0.6605407	0.2587461	0.3543712	0.0496879	0.0144538
-1.7	0.0549526	0.3927089	0.6649685	0.2674612	0.3870357	0.0520769	0.0145972
-1.6	0.0538831	0.3593026	0.6605916	0.2749955	0.4194518	0.0545239	0.0147378
-1.5	0.0527776	0.326384	0.6476396	0.2811918	0.4507861	0.0570237	0.0148753
-1.4	0.0516403	0.2945517	0.6267787	0.2859159	0.4801147	0.0595702	0.0150097
-1.3	0.0504753	0.264262	0.5990401	0.2890625	0.5064672	0.0621562	0.0151407
-1.2	0.0492868	0.2358366	0.5657159	0.2905599	0.5288852	0.0647739	0.0152682
-1.1	0.0480788	0.2094776	0.5282386	0.2903735	0.5464887	0.0674145	0.0153921
-1.0	0.0468554	0.1852862	0.4880627	0.2885077	0.5585451	0.0700682	0.0155123
-0.9	0.0456204	0.1632817	0.4465631	0.2850053	0.5645315	0.0727243	0.0156285
-0.8	0.0443774	0.1434208	0.4049592	0.279946	0.5641821	0.0753715	0.0157407
-0.7	0.0431299	0.1256145	0.3642685	0.2734418	0.5575125	0.0779973	0.0158487
-0.6	0.0418814	0.1097424	0.325287	0.2656327	0.5448184	0.0805888	0.0159523
-0.5	0.040635	0.0956654	0.2885923	0.2566798	0.5266483	0.0831324	0.0160516
-0.4	0.0393937	0.0832345	0.2545624	0.246759	0.5037541	0.0856137	0.0161462
-0.3	0.0381602	0.0722982	0.2234038	0.2360541	0.4770278	0.0880183	0.0162362
-0.2	0.0369371	0.062708	0.195184	0.2247503	0.4474325	0.0903313	0.0163214
-0.1	0.0357268	0.0543216	0.1698636	0.2130282	0.4159358	0.0925377	0.0164016
0.0	0.0345315	0.0470056	0.1473259	0.2010591	0.3834538	0.0946228	0.0164769
0.1	0.0333531	0.0406367	0.127402	0.189001	0.3508079	0.0965719	0.0165471
0.2	0.0321935	0.0351021	0.1098918	0.1769957	0.3186971	0.0983711	0.016612
0.3	0.0310542	0.0303	0.0945797	0.1651672	0.2876855	0.1000068	0.0166717
0.4	0.0299366	0.0261389	0.0812471	0.1536209	0.2582009	0.1014666	0.016726
0.5	0.0288419	0.0225375	0.0696804	0.1424433	0.2305439	0.1027391	0.0167748
0.6	0.0277712	0.0194235	0.059677	0.1317032	0.2049017	0.103814	0.0168182
0.7	0.0267255	0.0167333	0.0510487	0.1214524	0.1813661	0.1046826	0.016856
0.8	0.0257054	0.0144108	0.0436232	0.1117276	0.1599521	0.1053377	0.0168882
0.9	0.0247115	0.0124071	0.0372452	0.102552	0.1406154	0.1057739	0.0169148
1.0	0.0237443	0.0106793	0.031776	0.093937	0.1232688	0.1059875	0.0169356
1.1	0.0228041	0.0091902	0.0270927	0.0858842	0.1077956	0.1059767	0.0169508
1.2	0.0218911	0.0079073	0.0230871	0.0783868	0.0940614	0.1057416	0.0169602
1.3	0.0210056	0.0068023	0.0196646	0.0714318	0.0819226	0.1052841	0.0169638
1.4	0.0201473	0.005851	0.0167428	0.0650009	0.0712334	0.1046082	0.0169617
1.5	0.0193164	0.0050322	0.0142504	0.0590721	0.0618506	0.1037194	0.0169539
1.6	0.0185125	0.0043275	0.0121256	0.0536206	0.0536372	0.1026251	0.0169403
1.7	0.0177356	0.0037211	0.0103151	0.04862	0.0464645	0.1013341	0.016921
1.8	0.0169852	0.0031995	0.0087731	0.0440428	0.0402135	0.0998569	0.016896
1.9	0.0162611	0.0027508	0.0074603	0.0398615	0.0347755	0.0982049	0.0168653
2.0	0.0155627	0.0023649	0.006343	0.0360483	0.0300519	0.0963907	0.016829
2.1	0.0148897	0.0020331	0.0053923	0.0325764	0.0259543	0.0944279	0.0167871
2.2	0.0142416	0.0017477	0.0045836	0.0294196	0.0224037	0.0923305	0.0167397
2.3	0.0136178	0.0015024	0.0038959	0.0265532	0.0193302	0.0901132	0.0166869
2.4	0.0130178	0.0012914	0.0033111	0.0239533	0.0166719	0.0877907	0.0166287
2.5	0.012441	0.0011101	0.0028139	0.0215976	0.0143743	0.0853781	0.0165652
2.6	0.0118868	0.0009541	0.0023912	0.0194651	0.0123899	0.0828902	0.0164964
2.7	0.0113545	0.0008201	0.0020319	0.0175364	0.0106767	0.0803414	0.0164225
2.8	0.0108437	0.0007049	0.0017265	0.0157932	0.0091984	0.077746	0.0163436
2.9	0.0103535	0.0006059	0.001467	0.0142188	0.0079234	0.0751176	0.0162597
3.0	0.0098835	0.0005207	0.0012464	0.0127977	0.006824	0.072469	0.016171

Figure 10. Calculating the projected information generated by an exam consisting of the
seven example items.

ITEM	A2C	C15	B1B	C2A	C18B	A7	A5A	
SLOPE	0.29781	0.8911	0.95936	0.6343	0.88444	0.38305	0.15323	
DIFF	-3.26838	-2.53874	-1.69971	-1.16104	-0.85548	1.04519	1.31353	
ABILITY LEVEL								TEST INFO
-3.0	0.0637843	0.5090076	0.2543733	0.1236716	0.0831797	0.0264969	0.0125633	1.073076472
-2.9	0.0635251	0.5328087	0.288387	0.1340341	0.0954862	0.0280265	0.0127294	
-2.8	0.0631867	0.5518155	0.3250672	0.1448758	0.1094041	0.0296279	0.0128945	
-2.7	0.0627707	0.565233	0.3640368	0.1561413	0.1250762	0.0313023	0.0130584	
-2.6	0.0622792	0.5724744	0.4047197	0.1677581	0.1426357	0.0330505	0.0132209	
-2.5	0.0617147	0.5732141	0.446321	0.1796353	0.1621965	0.0348734	0.0133819	

Figure 11. Test information curve generated by the seven example items.
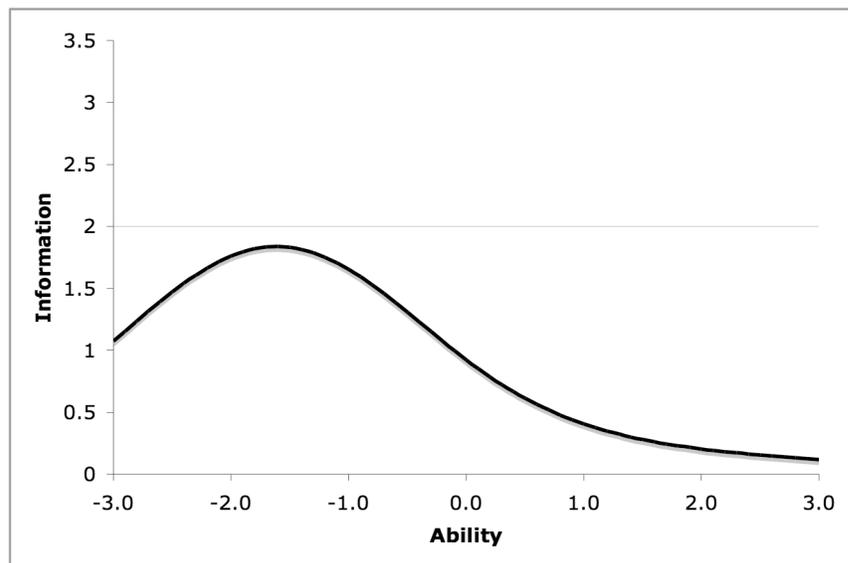
ability levels from -3.0 to +3.0 standard deviations, and the test information column
simultaneously (see Figure 11 for the test information curve created by the seven
example items).

For each of the four embedded measures and for the instrument as a whole, the
resulting hypothetical test information curve needed to be smooth with a maximum

occurring around -0.25 standard deviations (due to the projected ability level of the sample). Each curve was also checked to assure that the created instrument would generate adequate information over the entire range of teacher ability to be measured (-2.25 to +1.75 standard deviations were expected to exist among the sample). To assure that the estimated internal reliability of the measure was at least 0.7 for the desired range of abilities to be assessed, each test information curve needed to be above 2.0 units (units of information = slope of the test characteristic curve) across the designated domain. That is, each instrument should be able to reliability discriminate (at a level of 0.7 or higher) among teachers in the ability range where its test information curve is at least 2.0 units high. For example, the curve illustrated in Figure 11 would not be expected to reliability discriminate (at a level of 0.7 or higher) among any teachers since it does not generate at least 2.0 units of information for any ability level. The expected range of reliable discrimination, as identified from the test information curve, must coincide with the range of abilities expected in the sample. Figure 12 shows an example of a hypothetical test information curve that would be ideal, compared to the example test information curve from Figure 11.

If any of these stipulations were not met for the information curves generated for each of the four embedded measures and for the overall instrument, items were replaced and/or added, and the process outlined above was repeated. Item selection concluded once all five test information curves produced satisfactory results, using the least amount of items possible (see Figures 13-17 for individual test information curves). The resulting 51-item instrument was then checked for content validity. Two mathematics education

experts verified item coding and item selection in terms of the content to be addressed by

the embedded measures and the overall instrument.

Figure 12. Example ideal test information curve for the instrument under development, as compared to the test information curve generated by the seven example items.
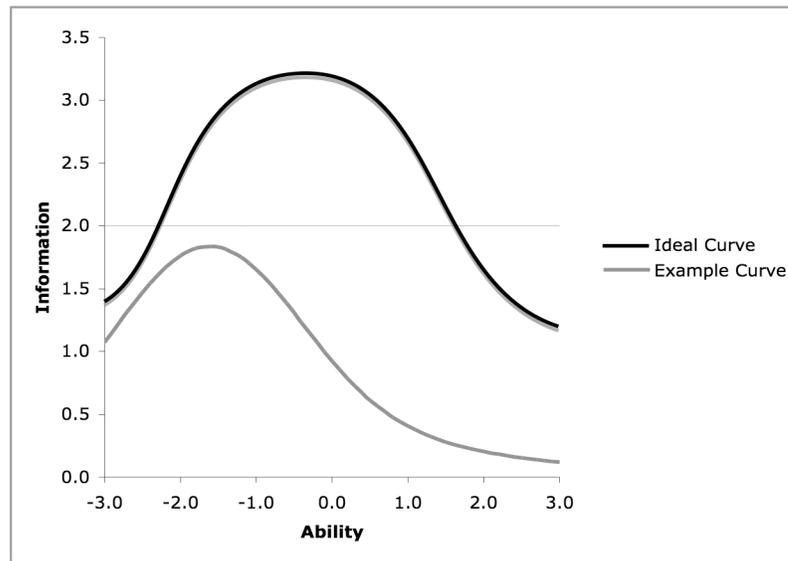
Figure 13. Test information curve generated by the 51 total items.
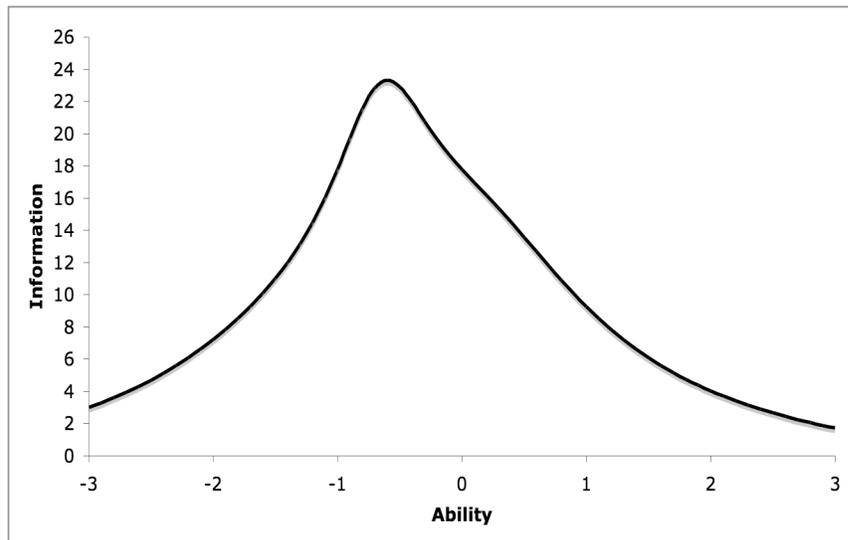
Figure 14. Test information curve generated by the 31 items that address common content knowledge.

Figure 15. Test information curve generated by the 20 items that address specialized content knowledge.

Figure 16. Test information curve generated by the 29 items that address knowledge of numbers.
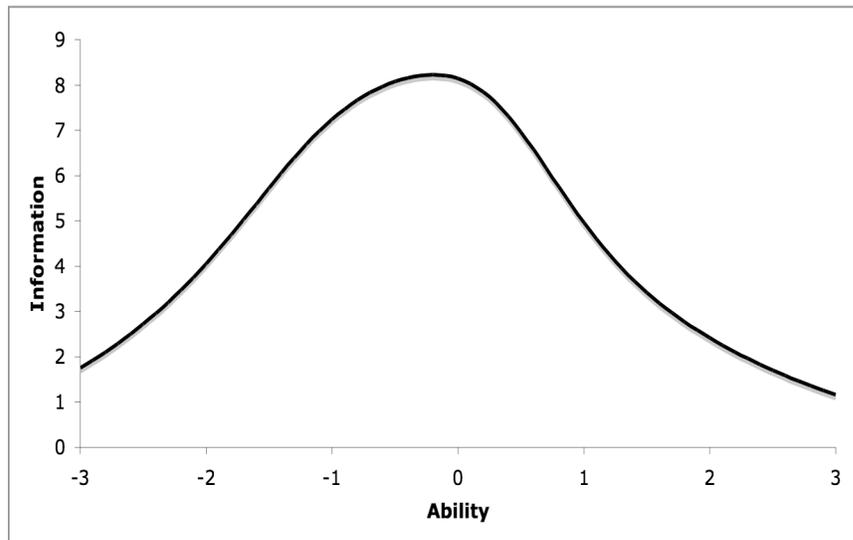
Figure 17. Test information curve generated by the 23 items that address knowledge of equations and functions.
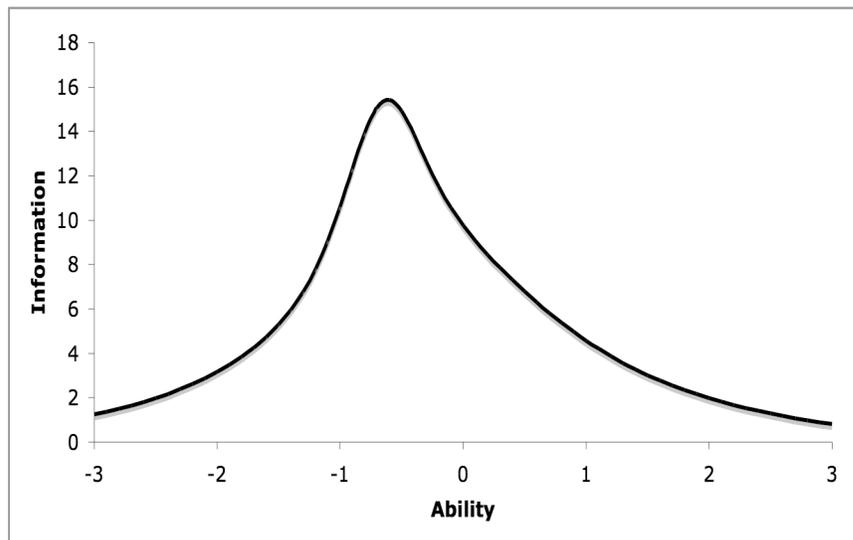
Methods

Instrument Administration and Scoring

On August 31, 2006, the instrument was administered to each section of Math 130 during its daily class meeting time. All Math 130 students present across three course sections ($n = 68$) were asked to complete the instrument developed to measure both common and specialized content knowledge of prerequisite algebra concepts. Students were informed that they were being tested as part of a doctoral dissertation study, but not explicitly that they would again be tested at the end of the semester. Participation was voluntary and results were confidential (only the last four digits of students' social security numbers were used as a matching scheme). Students were told that since the purpose of this study was to assure that teacher preparation courses are preparing students to be successful teachers, they should not make any random guesses on the instrument; instead, any item for which they felt they could not make an educated guess should be left blank. Although students were not given incentives to participate and completion of the instrument had no bearing on students' grades, all but one present student participated in the study. After signing a written content form, willing participants were allowed the entire 50-minute class period to complete the instrument. The majority of students worked on the instrument for 35-50 minutes.

All completed copies of the instrument were scored using Excel. Each student received a total score for the overall number of correct answers provided, in addition to a score showing the number of correct answers provided for each of the four individual constructs (common content knowledge, specialized content knowledge, content

knowledge of numbers, and content knowledge of equations and functions). Pretest

scores were used to calculate a Cronbach's Alpha to test the internal reliability of the

overall instrument and each of the four embedded measures. Using the criteria of $\alpha \geq .70$

needed for sample sizes of 60 or more, as suggested by the LMT, all embedded measures,

as well as the overall instrument, were shown to be reliable (see Table 1 for individual

reliability results).

Table 1. Internal Reliability Results.

	Cronbach's Alpha
Total Instrument	.8376 ($n = 68$)
Common Content Knowledge	.7573 ($n = 68$)
Specialized Content Knowledge	.7279 ($n = 68$)
Content Knowledge of Numbers	.7196 ($n = 68$)
Content Knowledge of Equations/Functions	.7413 ($n = 68$)

The instrument was administered a second time to all Math 130 students, on

November 28, 2006 ($n = 54$). Students were reminded of the purpose of the testing and,

only at this time, informed of the pretest-posttest nature of the study. All present students

participated in this administration and, again, most worked for 35-50 minutes. Since the

instrument used in this study had only one form that was administered twice (as both

pretest and posttest), the second administration date was carefully chosen to ensure that

test dates were separated by at least three months. When identical forms are completed

multiple times, test-retest effects can occur. However, the LMT researchers suggest that

if three or more months pass between two administrations of an exam, these effects will

be minimal and not substantially affect the outcome of the study.

A total of 48 students completed both administrations of the exam and were therefore included in the sample used to answer the research questions of this study. Every participant received five scores for both the pretest and posttest administrations, resulting in a total of ten raw scores. Recall that this instrument was designed so that an average preservice teacher (ability level of -0.25 SD) would correctly answer 50% of the items; and, the results of this study did reflect this percentage. Because of this design, the LMT researchers discourage reporting raw scores and/or percentages because they may mislead the public about teachers' overall level of content knowledge. Therefore, all raw scores were standardized according to the statistics calculated from the pretest scores for each measure (see Table 2 for statistics). Once all raw pretest scores were standardized to z-scores, a raw score to z-score conversion table was constructed for each measure. These tables were then used to standardize the raw scores resulting from each posttest.

Table 2. Statistics from Pretest Scores.

	Pretest	
	Mean	Standard Deviation
Total Instrument	$\bar{x}_{pre_T} = 23.5625$	$s_{pre_T} = 7.906$
Content Knowledge of Numbers	$\bar{x}_{pre_N} = 12.8125$	$s_{pre_N} = 4.4895$
Content Knowledge of Equations/Functions	$\bar{x}_{pre_E} = 4.1599$	$s_{pre_E} = 4.1599$
Common Content Knowledge	$\bar{x}_{pre_C} = 14.7500$	$s_{pre_C} = 5.0634$
Specialized Content Knowledge	$\bar{x}_{pre_S} = 8.8125$	$s_{pre_S} = 3.6241$

Data Analysis

Standardized instrument scores were examined using a variety of statistical analyses to address the range of research questions presented in this study. Variables

defined for growth analysis are listed below in Table 3. All data analysis was completed

using *SPSS 11.0 for Mac OS X*. Results for each investigation described in this section

are presented in Chapter 4 (pp. 94-115).

Table 3. Variables Defined.

Knowledge Type	Variable Measuring Growth
Total Content Knowledge	$x_{T_i} = \dfrac{post_{T_i} - \bar{x}_{pre_T}}{s_{pre_T}} - \dfrac{pre_{T_i} - \bar{x}_{pre_T}}{s_{pre_T}} = \dfrac{post_{T_i} - pre_{T_i}}{s_{pre_T}}$
Content Knowledge of Numbers	$x_{N_i} = \dfrac{post_{N_i} - \bar{x}_{pre_N}}{s_{pre_N}} - \dfrac{pre_{N_i} - \bar{x}_{pre_N}}{s_{pre_N}} = \dfrac{post_{N_i} - pre_{N_i}}{s_{pre_N}}$
Content Knowledge of Equations/Functions	$x_{E_i} = \dfrac{post_{E_i} - \bar{x}_{pre_E}}{s_{pre_E}} - \dfrac{pre_{E_i} - \bar{x}_{pre_E}}{s_{pre_E}} = \dfrac{post_{E_i} - pre_{E_i}}{s_{pre_E}}$
Common Content Knowledge	$x_{C_i} = \dfrac{post_{C_i} - \bar{x}_{pre_C}}{s_{pre_C}} - \dfrac{pre_{C_i} - \bar{x}_{pre_C}}{s_{pre_C}} = \dfrac{post_{C_i} - pre_{C_i}}{s_{pre_C}}$
Specialized Content Knowledge	$x_{S_i} = \dfrac{post_{S_i} - \bar{x}_{pre_S}}{s_{pre_S}} - \dfrac{pre_{S_i} - \bar{x}_{pre_S}}{s_{pre_S}} = \dfrac{post_{S_i} - pre_{S_i}}{s_{pre_S}}$

Note: ($n = 48$) ($i = 1, 2,…, 48$).

<u>Effects of Math 130 on Mathematical Content Knowledge.</u> To answer the first

research question, which explores gains in the mathematical content knowledge of

prerequisite algebra concepts, a matched pairs *t*-test (*t*) was used to compare pretest and

posttest total scores within the single sample. The variable, x_{T_i}, was defined (see Table 3)

to calculate the difference in each student's standardized pretest and posttest total scores

($i = 1, 2,…,48$). The hypothesis claiming that the true mean difference in standardized

pretest and posttest total scores, μ_T, is equal to zero was tested against a two-sided

hypothesis for a non-zero population mean difference.

Effects of Math 130 on Prerequisite Algebra Concepts and Type of Content

Knowledge. Similarly, the second and third research questions address gains in

knowledge of individual prerequisite algebra constructs (numbers and

equations/functions) and each type of content knowledge (common versus specialized).

To address these questions, variables x_{N_i}, x_{E_i}, x_{C_i}, and x_{S_i} were similarly defined (see

Table 3) and the analysis process outlined above was repeated for each of the four

embedded construct measures.

Relationship Between Common and Specialized Content Knowledge. The fourth

research question focuses on the relationship between common and specialized content

knowledge. Correlation analysis of the standardized scores for common and specialized

content knowledge obtained from the second administration of the instrument was

utilized to investigate the relationship between the two types of content knowledge of

prerequisite algebra concepts for preservice teachers.

Analysis of Performance on Individual Items. The last research question seeks to

explore items missed by more or less students than what would be predicted by the item's

difficulty level. Therefore, difficulty information for the items was used to create a one-

parameter linear model to predict the number of participants that would incorrectly

answer each item. This model was built using linear regression on each item's difficulty

level and the number of incorrect responses recorded for the item on the posttest. For

each item on the instrument, a residual (observed number of incorrect answers – number

of incorrect answers predicted by the regression model) was calculated.

The set of residuals was analyzed to identify items outside of the overall pattern. These items were identified and grouped according to the sign of their residual values. Those with positive residual values represent the group of items missed by more students than what was predicted by the linear model. Those with negative residual values, on the other hand, represent the group of items missed by less students than predicted. Items in each group were analyzed for the content and type of knowledge addressed, and the two groups were assessed for any existing trends or patterns.

To determine whether items with large residuals on the posttest also had large residuals on the pretest, a second linear regression was performed using each item's difficulty level to predict the number of incorrect responses given for the item on the first administration of the instrument (pretest). The residuals of each item in the two different models were compared and examined. It should be noted that at the time of this analysis, three items included on the instrument had not yet been piloted by the LMT. Therefore, only the 48 items for which difficulty information was accessible were used in the calculation of these models.

<div align="center">Assumptions and Limitations</div>

Assumptions

Prerequisite Algebra Concepts. This study relies on the assumption that there are nine unique concepts that should be considered prerequisite to a learner's first algebra course (see p. 2), and that these nine concepts can be categorized into two constructs, namely number concepts and equation/function concepts (see pp. 71-72).

The Theoretical Framework Appropriately Categorizes Knowledge for Teaching

Mathematics. The Theoretical Framework for Knowledge for Teaching Mathematics (see

pp. 4-5) was developed for this study and guided the development of the instrument used

in this study.

Limitations

Incoming Student Knowledge. Students at Montana State University are required

to demonstrate a certain minimum level of prerequisite knowledge prior to taking Math

130 (see pp. 130-131 for course prerequisite details). Since these standards may differ

from those imposed at other universities, some limitations exist regarding the populations

to which results of this study can be generalized. It should also be noted that the course

prerequisites were not strictly enforced during the semester in which the data for this

study was collected. Furthermore, prerequisites only define a minimum comprehension

level and therefore do no clearly identify the true variability that may exist in the

knowledge students have upon entering this course.

Pre-Experimental One-Group Pretest-Posttest Design. There was no control group

that was tested during this study. Therefore, as with all one-group pretest-posttest

designs, it cannot be conclusively determined if any growth in student knowledge was in

fact a result of the students having taken the Math 130 course. Many factors may

influence (positively or adversely) the results of this type of a study; hence, there may be

additional reasons for observed knowledge growth, such as a testing effect.

Blank Responses. Although the students all appeared to have enough time to complete both the pretest and the posttest administrations of the instrument, it is possible that some blank responses resulted from lack of time or motivation and not necessarily from a lack of understanding. Since blank responses were treated the same as incorrect responses, it was necessary to analyze the number of blank responses reported for each item and each student on both of the administrations.

No significant linear relationship appeared between the number of blank responses for and the order of the questions on the posttest ($p < .451$, $\alpha = .05$). However, a significant increase in the number of blank responses with respect to the order of the questions was observed on the pretest ($p = .001$, $\alpha = .05$). Although this result implies that student may have run out of time when completing the pretest, the slope of the linear regression model built to predict the number of blank responses for each item on the pretest based on question order was very small ($\beta_1 = .0686$).

This increase was only observed for the second half of the posttest, which included a mixture of common and specialized content knowledge items, but only number concepts items (as the first half of the exam was comprised of equation/function items). Consequently, if student performance on the pretest was affected by insufficient time, the results of the matched pairs t-tests (which calculated posttest scores − pretest scores), exploring gains in total knowledge and number concepts knowledge, could be influenced. To examine this possibility, the effect of the difference in blank responses per student (number of blank responses on posttest − number of blank responses on pretest) on difference in incorrect responses per student (number of incorrect responses on

posttest – number of incorrect responses on pretest) was investigated, first using all 51 items and again only using the 29 number concepts items. This effect was found to be insignificant for both total scores (p = .368, α = .05) and number concept scores (p = .187, α = .05). Therefore, the matched pairs t-tests used in this study did not appear to be affected by the observed increase in blank responses for the latter items of the pretest.

For the fifth and last research question, the simple linear regression used to identify interesting items was built using the number of incorrect answers provided for each item on the posttest (where order did not have a significant effect on the number of blank responses). Pretest results were only used to build a second simple linear regression model for qualitative comparisons.

CHAPTER 4

RESULTS

<u>Introduction</u>

Data collection and analysis for this study was designed to focus on several research questions, all with respect to an undergraduate first-semester elementary education mathematics content course (Math 130). This chapter presents the results of the data analysis, divided into three sections.

The first section of data analysis contains the results of the five matched pairs *t*-tests which investigate the effects of Math 130 on preservice elementary teachers' mathematical content knowledge, common content knowledge, and specialized content knowledge of prerequisite algebra concepts, as well as content knowledge of number concepts and equation/function concepts.

This study also explores the relationship between preservice teacher's common content knowledge and specialized content knowledge of prerequisite algebra concepts. Results of the correlation analysis applied to posttest common and specialized content knowledge scores is located in the second section of data analysis.

The final section of data analysis searches for patterns that exist among items missed by more or less preservice elementary teachers than predicted on the administered instrument. Items' residuals are analyzed after a discussion on the building of a linear model used to predict the number of participants to incorrectly answer each item on the posttest based upon its difficulty level. Items with residuals falling outside the overall

pattern were identified and analyzed. Note that due to LMT restrictions, specific items from the instrument cannot be published. Therefore, items of interest will only be discussed in terms of the content and type of knowledge they address.

<div align="center">Matched Pairs <i>T</i>-tests</div>

Mathematical Content Knowledge of Prerequisite Algebra Concepts

1. What effects does this course have on preservice teachers' mathematical content knowledge of prerequisite algebra concepts?

To examine growth in the mathematical content knowledge of prerequisite algebra concepts, differences in standardized pretest and posttest total scores were examined. All total scores were standardized according to the mean pretest total score and the standard deviation for all pretest total scores (see Tables 2 and 3, pp. 86-87 in Chapter 3, for statistics and variable definitions).

The mean standardized difference in pretest and posttest total scores within the sample was $\bar{x}_T = \frac{1}{48} \sum_{i}^{48} x_{T_i} = .6430$, indicating that students' total mathematical content knowledge of prerequisite algebra concepts improved an average of .6430 pretest standard deviations. Therefore, the students correctly answered an average of 5.08 more of the total 51 items on the posttest versus the pretest. A matched pairs two-sided t-test using the sample mean standardized difference in total scores was significant, implying that the true mean standardized total score difference is believed to be non-zero.

Table 4. Total Scores.

	Standardized Difference in Total Scores
t-test	$t_T = 9.248$
p-value	$p < .001^*$
Sample Mean Difference	$\bar{x}_T = .6430$
95% Confidence Interval for True Mean Difference	$\mu_T \in (.5031, .7828)$

* Result was significant using $\alpha = .05$.

These results show with 95% confidence that the true mean standardized difference in total posttest and pretest scores lies within the interval (.5031, .7828), indicating that students improve an average of .5031 to .7828 pretest standard deviations on their total mathematical content knowledge of prerequisite algebra concepts upon completion of Math 130. This standardized increase translates to students answering an average of 3.98 to 6.19 more of the 51 total items correctly on the posttest versus the pretest. True mean improvement on the total instrument is therefore believed to fall in the range of 7.8% to 12.14%.

Knowledge of Individual Prerequisite Algebra Constructs

2. What effects does this course have on preservice teachers' mathematical content knowledge of individual prerequisite algebra constructs (number concepts and equation/function concepts)?

To address individual prerequisite algebra constructs, differences in standardized pretest and posttest number scores and differences in standardized pretest and posttest equation/function scores were examined. All number scores were standardized according to the mean pretest number score and the standard deviation for all pretest number scores. Similarly, all equation/function scores were standardized to pretest equation/function statistics (see Tables 2 and 3 in Chapter 3 for statistics and variable definitions).

Number Concepts. The mean standardized difference in pretest and posttest number scores within the sample was $\bar{x}_N = \frac{1}{48}\sum_i^{48} x_{N_i} = .7889$, indicating that students' mathematical content knowledge of number concepts improved an average of .7889 pretest standard deviations. Therefore, the students correctly answered an average of 3.54 more of the 29 number items on the posttest versus the pretest. A matched pairs two-sided *t*-test using the sample mean standardized difference in number scores was significant, implying that the true mean standardized number score difference is believed to be non-zero.

These results show with 95% confidence that the true mean standardized difference in number posttest and pretest scores lies within the interval (.5857, .9921), indicating that students improve an average of .857 to .9921 pretest standard deviations on their mathematical content knowledge of number concepts upon completion of Math 130. This standardized increase translates to students answering an average of 2.63 to 4.45 more of the 29 number items correctly on the posttest versus the pretest. True mean

improvement in terms of number concepts is therefore believed to fall in the range of 9.07% to 15.36%.

Table 5. Number Scores.

	Standardized Difference in Number Scores
t-test	$t_N = 7.810$
p-value	$p < .001^*$
Sample Mean Difference	$\bar{x}_N = .7889$
95% Confidence Interval for True Mean Difference	$\mu_N \in (.5857, .9921)$

* Result was significant using $\alpha = .05$.

Equation/Function Concepts. The mean standardized difference in pretest and posttest equation/function scores within the sample was $\bar{x}_E = \frac{1}{48} \sum_{i}^{48} x_{E_i} = .3906$, indicating that students' mathematical content knowledge of equation/function concepts improved an average of .3906 pretest standard deviations. Therefore, the students correctly answered an average of 1.63 more of the 23 equation/function items on the posttest versus the pretest. A matched pairs two-sided *t*-test using the sample mean standardized difference in equation/function scores was significant, implying that the true mean standardized equation/function score difference is believed to be non-zero.

Table 6. Equation/Function Scores.

	Standardized Difference in Equation/Function Scores
t-test	$t_E = 4.704$
p-value	$p < .001^*$
Sample Mean Difference	$\bar{x}_E = .3906$
95% Confidence Interval for True Mean Difference	$\mu_E \in (.2236, .5577)$

* Result was significant using $\alpha = .05$.

These results show with 95% confidence that the true mean standardized difference in equation/function posttest and pretest scores lies within the interval (.2236, .5577), indicating that students improve an average of .2236 to .5577 pretest standard deviations on their mathematical content knowledge of equation/function concepts upon completion of Math 130. This standardized increase translates to students answering an average of .93 to 2.32 more of the 23 equation/function items correctly on the posttest versus the pretest. True mean improvement in terms of equation/function concepts is therefore believed to fall in the range of 4.04% to 10.09%.

Common and Specialized Content Knowledge
of Prerequisite Algebra Constructs

3. What effects does this course have on preservice teachers' common content knowledge and specialized content knowledge of prerequisite algebra concepts?

To investigate individual types of content knowledge, differences in standardized pretest and posttest common content knowledge scores and differences in standardized pretest and posttest specialized content knowledge scores were examined. All common content knowledge scores were standardized according to the mean pretest common content knowledge score and the standard deviation for all pretest common content knowledge scores. Similarly, all specialized content knowledge scores were standardized to pretest standardized statistics (see Tables 2 and 3 in Chapter 3 for statistics and variable definitions).

Common Content Knowledge. The mean standardized difference in pretest and posttest common content knowledge scores within the sample was $\bar{x}_C = \frac{1}{48}\sum_{i}^{48} x_{C_i} = .5431$, indicating that students' common content knowledge of prerequisite algebra concepts improved an average of .5431 pretest standard deviations. Therefore, the students correctly answered an average of 2.75 more of the 31 common content knowledge items on the posttest versus the pretest. A matched pairs two-sided t-test using the sample mean standardized difference in common content knowledge scores was significant, implying that the true mean standardized common content knowledge score difference is believed to be non-zero.

Table 7. Common Content Knowledge Scores.

	Standardized Difference in Common Content Knowledge Scores
t-test	$t_C = 6.192$
p-value	$p < .001^*$
Sample Mean Difference	$\bar{x}_C = .5431$
95% Confidence Interval for True Mean Difference	$\mu_C \in (.3667, .7196)$

* Result was significant using $\alpha = .05$.

These results show with 95% confidence that the true mean standardized difference in common content knowledge posttest and pretest scores lies within the interval (.3667, .7196), indicating that students improve an average of .3667 to .7196 pretest standard deviations on their common content knowledge of prerequisite algebra concepts upon completion of Math 130. This standardized increase translates to students answering an average of 1.86 to 3.64 more of the 31 common content knowledge items correctly on the posttest versus the pretest. True mean improvement in common content knowledge is therefore believed to fall in the range of 6.0% to 11.74%.

Specialized Content Knowledge. The mean standardized difference in pretest and posttest specialized content knowledge scores within the sample was $\bar{x}_S = \frac{1}{48} \sum_{i}^{48} x_{S_i} = .6438$, indicating that students' specialized content knowledge of prerequisite algebra concepts improved an average of .6438 pretest standard deviations.

Therefore, the students correctly answered an average of 2.33 more of the 20 specialized content knowledge items on the posttest versus the pretest. A matched pairs two-sided *t*-test using the sample mean standardized difference in specialized content knowledge scores was significant, implying that the true mean standardized specialized content knowledge score difference is believed to be non-zero.

Table 8. Specialized Content Knowledge Scores.

	Standardized Difference in Specialized Content Knowledge Scores
t-test	$t_S = 5.198$
p-value	$p < .001^*$
Sample Mean Difference	$\bar{x}_S = .6438$
95% Confidence Interval for True Mean Difference	$\mu_S \in (.3946, .8930)$

* Result was significant using $\alpha = .05$.

These results show with 95% confidence that the true mean standardized difference in specialized content knowledge posttest and pretest scores lies within the interval (.3946, .8930), indicating that students improve an average of .3946 to .8930 pretest standard deviations on their specialized content knowledge of prerequisite algebra concepts as result of taking Math 130. This standardized increase translates to students answering an average of 1.43 to 3.24 more of the 20 specialized content knowledge items correctly on the posttest versus the pretest. True mean improvement in specialized content knowledge is therefore believed to fall in the range of 7.15% to 16.18%.

Correlation Analysis

4. What relationship, if any, exists between preservice elementary teachers'
   common and specialized content knowledge of prerequisite algebra concepts?

To investigate the possible relationship between common and specialized content knowledge, correlation analysis was applied to the total raw common and specialized content knowledge scores attained during the second administration of the instrument. A resulting Pearson's correlation coefficient of .716 shows a statistically significant positive relationship ($p < .001$, $\alpha = .05$) between preservice elementary teachers common and specialized content knowledge of prerequisite algebra concepts.

Linear Regression

5. What patterns, if any, exist among items missed by more or less preservice
   elementary teachers than predicted on the instrument measuring mathematical
   content knowledge of prerequisite algebra concepts?

First, a linear model was built from the results of the second administration of the instrument to predict the number of preservice elementary teachers that would incorrectly answer each item on the posttest. Residuals were then analyzed and items that were missed by notably more or less participants than predicted by the model were identified and explored for any existing patterns in content area and/or knowledge type. A second

liner model was then built from the results of the first administration so that student performance on the isolated items, on the pretest, could also be investigated.

Building a Linear Model from Posttest Data

Items with higher difficulty levels will tend to have more students answering them incorrectly ($F = 66.945$, $df = 1, 46$, $p < .001$). In fact, an item's difficulty level alone was found to account for over 59% ($r^2 = .593$) of the variability in the number of participants that incorrectly answer an item on the posttest. Therefore, the researcher was able to construct a one-parameter linear model, $y_{wrong} = \beta_o + \beta_1 x_{diff} + \varepsilon \ (\varepsilon \sim N(0, \sigma^2))$, based on difficulty alone, to predict the number of participants that would incorrectly answer each item on the posttest. The resulting regression model, $\hat{y}_{wrong} = 23.665 + 8.658 x_{diff}$, with mean square error of 46.186, was used to identify items of interest (see Tables 9 and 10 for model details and Figure 18 for model fit).

Table 9. Posttest Simple Linear Model Summary.[ab]

	Model Statistics
R	$r = .770$
R Square	$r^2 = .593$

a. Predictors: (Constant), Difficulty.
b. Dependent Variable: Number of Wrong Answers on Posttest.

Table 10. Posttest Simple Linear Model Coefficients.[a]

| | Unstandardized Coefficients | | t | Sig. |
	B	Std. Error		
(Constant)	23.665	1.030	22.984	$p < .001^*$
Difficulty	8.658	1.058	8.182	$p < .001^*$

a. Dependent Variable: Number of Wrong Answers on Posttest.
  * Result was significant using $\alpha = .05$.

Figure 18. Posttest data and resulting linear regression model, with items of interest circled (see following discussion for item selection).
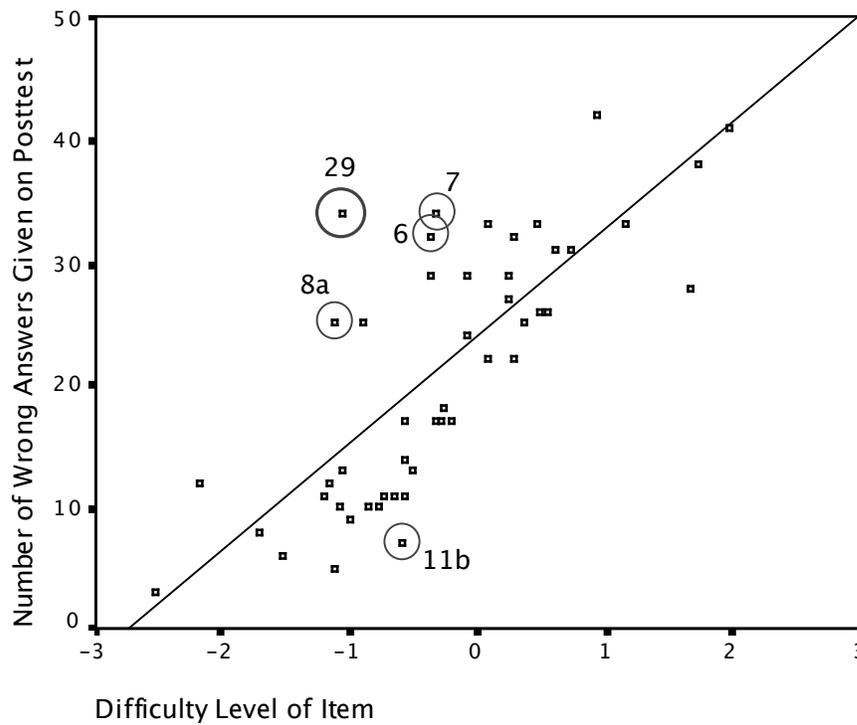
The pilot data suggested that significant linear relationships exist between the difficulty of an item and the number of incorrect answers provided for that item, as well

as between item slope and number of incorrect answers. Based on pilot results, originally

both difficulty and slope were used as predictors to create a two-parameter linear model,

$y_{wrong} = \beta_o + \beta_1 x_{diff} + \beta_2 x_{slope} + \varepsilon \, (\varepsilon \sim N(0, \sigma^2))$, to predict the number of participants that

would incorrectly answer each item on the instrument. However, the data collected

during the second administration of this study's instrument suggests that the slope of an

item is not significantly linearly related to the number of incorrect answers given for an

item ($p = .126$, $\alpha = .05$). In fact, when slope was removed from the linear model, mean

square error only slightly increased (to 46.186 from 44.788) and r^2 only slightly

decreased (from .614 to .593), showing a very small reduction in the predictive power of

the one-parameter linear model in comparison to the two-parameter model.

After the one-parameter linear model for the posttest was constructed, both the

normal quantile plot for the model (see Figure 19) and a scatterplot showing standardized

Normal P-P Plot of
Standardized Residuals

Figure 19. Normal quantile plot of posttest standardized residuals.
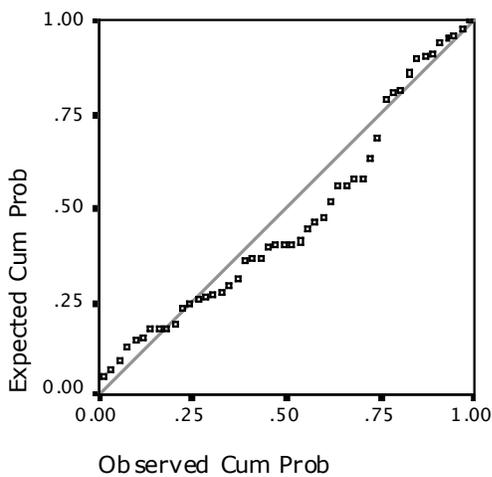
Dependent Variable: Number of Wrong Answers on Posttest.

predicted values plotted against standardized residuals (see Figure 20) were checked to assure linear model assumptions were all met. No indications of serious violations of linearity, constant variance, or normality of residuals, were found.

Figure 20. Scatterplot of posttest standardized residuals versus standardized predicted values.

Dependent Variable: Number of Wrong Answers on Posttest.
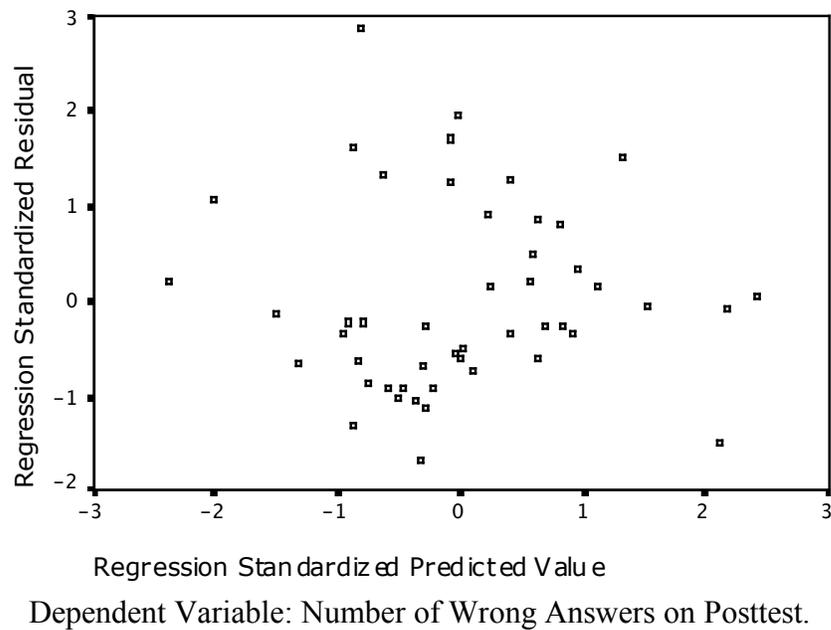
Isolating Items of Interest

Based on model, $\hat{y}_{wrong} = 23.665 + 8.658x_{diff}$ created from posttest data, the residuals (observed number of incorrect answers – number of incorrect answers predicted by the regression line) were calculated for each item. The set of residuals were then analyzed for outliers, by identifying residuals lying more than 1.5 times the interquartile range above the third quartile or below the first quartile of residuals. Residual values

ranged from -11.44 to 19.54, with an interquartile range of 8.8992 ($Q_3 - Q_1 = 4.3868 - (-4.5124) = 8.8992$). Only question #29 had a residual, $e_{29} = 19.54$ ($z_{e_{29}} = 2.88$), far enough from the overall pattern to be identified as an outlier (this item has been indicated by a thick, dark circle in Figure 18).

All residuals were then arranged in descending order according to their distances from zero, and five of the 51 items (10% of the total) whose residuals fell farthest from zero were isolated as "items of interest" (indicated by the five circles in Figure 18). See Table 11 for details regarding the five isolated, interesting items. The main ideas of these five items will be discussed later in general terms, however due to LMT regulations regarding non-released items, specifics entailed in the questions themselves cannot be discussed.

Table 11. Interesting Items.

	Q #29	Q #7	Q #6	Q #11b	Q #8a
difficulty	$x_{diff} = -1.063$	$x_{diff} = -0.3328$	$x_{diff} = -0.375$	$x_{diff} = -0.604$	$x_{diff} = -1.1232$
wrong (obs)	$y_{wrong} = 34$	$y_{wrong} = 34$	$y_{wrong} = 32$	$y_{wrong} = 7$	$y_{wrong} = 25$
wrong (pred)	$\hat{y}_{wrong} = 14.46$	$\hat{y}_{wrong} = 20.78$	$\hat{y}_{wrong} = 20.42$	$\hat{y}_{wrong} = 18.44$	$\hat{y}_{wrong} = 13.94$
residual$_{post}$	$e_{29} = 19.54$	$e_7 = 13.22$	$e_6 = 11.58$	$e_{11b} = -11.44$	$e_{8a} = 11.06$
std. residual	$z_{e_{29}} = 2.88$	$z_{e_7} = 1.94$	$z_{e_6} = 1.70$	$z_{e_{11b}} = -1.68$	$z_{e_{8a}} = 1.63$
knowledge	specialized	common	common	common	specialized
construct	numbers	equations/ functions	equations/ functions	equations/ functions	equations/ functions
content	proportions/ ratios	representing functions	writing functions	linear graphs	functions in context

Building a Linear Model from Pretest Data

To further investigate student performance on these five items identified as having residuals falling outside of the overall pattern, including the one aforementioned outlier, a second linear regression model was created using the pretest data. This one-parameter linear model, $y_{wrong} = \beta_o + \beta_1 x_{diff} + \varepsilon$ $(\varepsilon \sim N(0, \sigma^2))$, based on difficulty, predicts the number of participants that would incorrectly answer each item on the pretest (see Tables 12 and 13 for model details). Figure 21 shows a scatterplot of the pretest data and the fit of the resulting regression model, $\hat{y}_{wrong} = 28.747 + 9.005 x_{diff}$. Additionally, the five items identified earlier (questions #6, #7, #8a, #11b, and #29) have been marked on Figure 21, with circles.

Table 12. Pretest Simple Linear Model Summary.[ab]

	Model Statistics
R	$r = .803$
R Square	$r^2 = .645$

a. Predictors: (Constant), Difficulty.
b. Dependent Variable: Number of Wrong Answers on Pretest.

Table 13. Pretest Simple Linear Model Coefficients.[a]

	Unstandardized Coefficients		t	Sig.
	B	Std. Error		
(Constant)	28.474	.959	29.962	$p < .001^*$
Difficulty	9.005	.986	9.133	$p < .001^*$

a. Dependent Variable: Number of Wrong Answers on Pretest.
  * Result was significant using $\alpha = .05$.

Figure 21. Pretest data and resulting linear regression model.
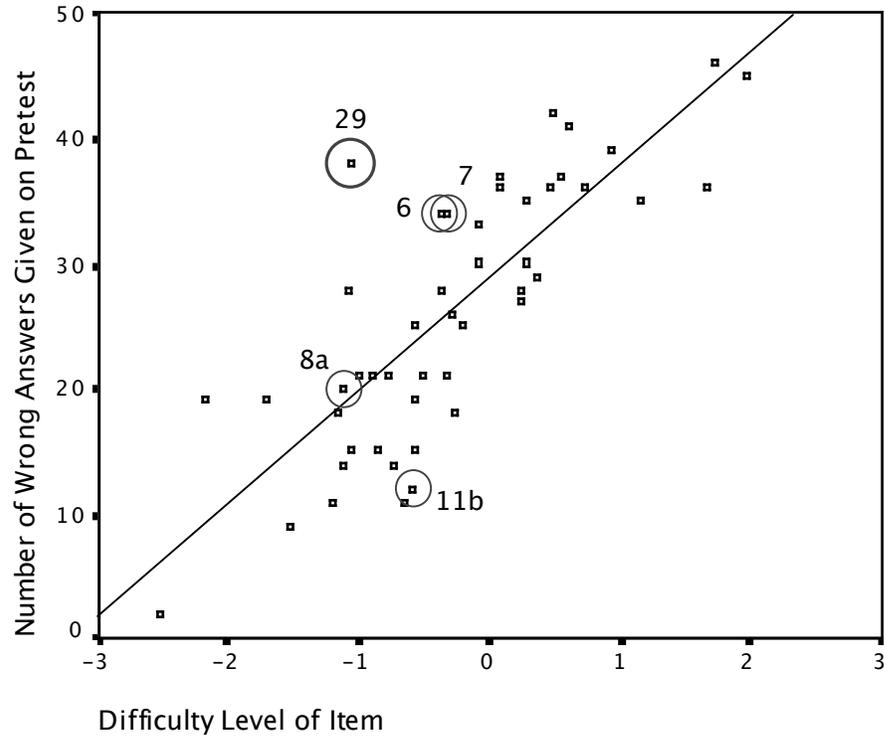
This model's normal quantile plot (see Figure 22) and a scatterplot showing standardized predicted values plotted against standardized residuals (see Figure 23) were checked to assure linear model assumptions were all met. No indications of serious violations of linearity, constant variance, or normality of residuals, were found. When the set of residuals was analyzed, question #29 was again the only item with a residual value identified as an outlier.

Normal P–P Plot of

Standardized Residuals

Figure 22. Normal quantile plot of pretest standardized residuals.
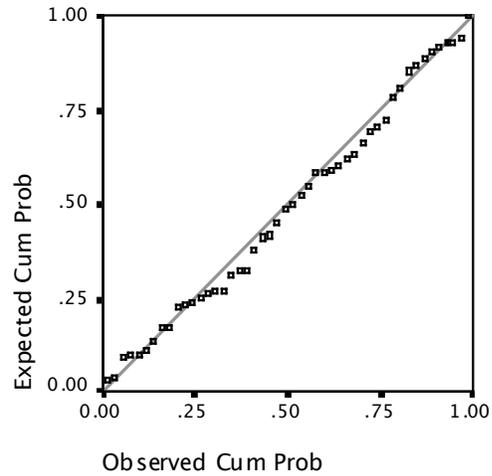
Dependent Variable: Number of Wrong Answers on Pretest.

Figure 23. Scatterplot of pretest standardized residuals versus standardized predicted values.

Regression Standardized Predicted Value
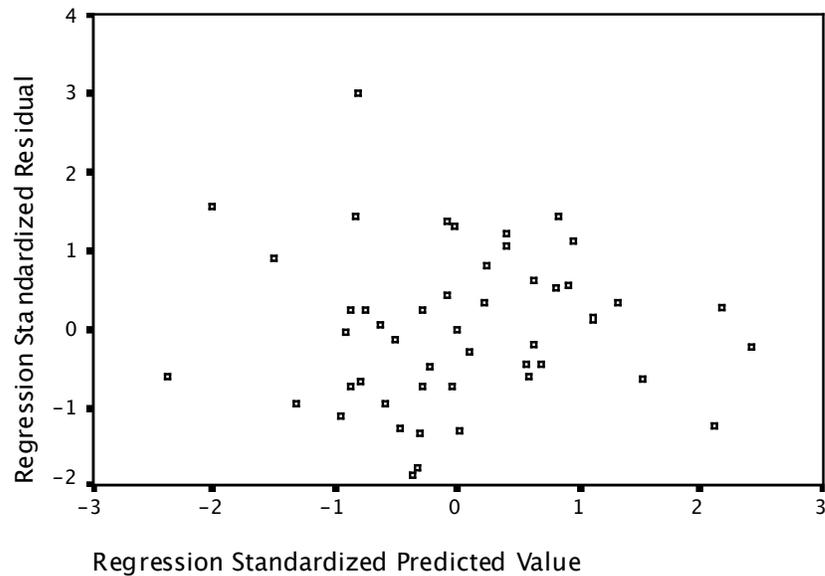
Dependent Variable: Number of Wrong Answers on Pretest.

Comparison of Pretest and Posttest Linear Models

Two simple linear regression models were built using the difficulty level of each item and the number of incorrect answers that were given by preservice teachers for each item on the pretest and posttest administrations of the instrument. When these two models were graphed simultaneously on the same axis, the resulting lines appeared to have roughly parallel slopes (m_{pre} = 9.005, m_{post} = 8.658) but moderately distinct y-intercepts (y-int$_{pre}$ = 28.747, y-int$_{post}$ = 23.665) (see Figure 24). To test if the difference that appeared between the two simple linear regression models was significant, a general linear model, $y_{wrong} = \beta_o + \beta_1 I_{pre} + \beta_2 x_{diff} + \beta_3 I_{pre} x_{diff} + \varepsilon$ ($\varepsilon \sim N(0, \sigma^2)$), was constructed using an indicator variable, I_{pre}, to denote if a given score resulted from a pretest or posttest (1 for pretest, 0 for posttest) (see Table 14 for model details).

Figure 24. Pretest and posttest simple linear regression models graphed simultaneously.

Difficulty Level of Item
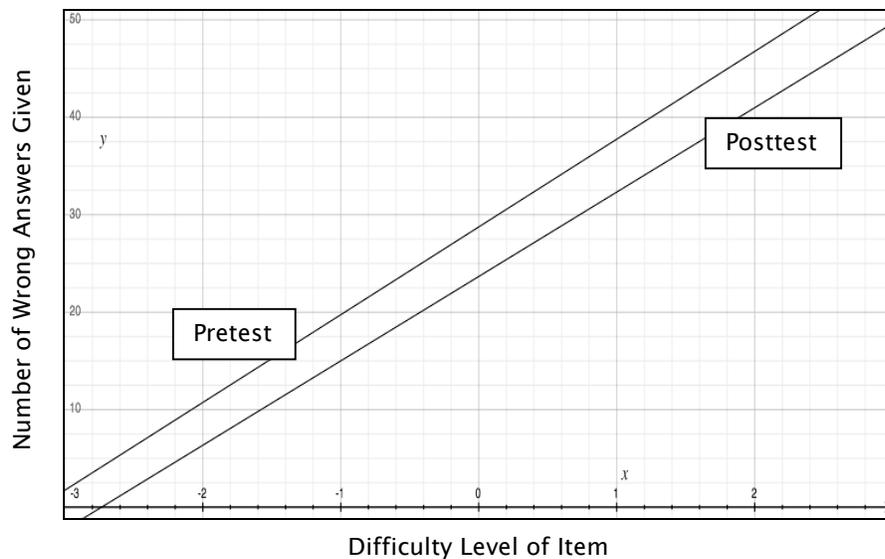
Table 14. General Linear Model Coefficients.[a]

	Unstandardized Coefficients		t	Sig.
	B	Std. Error		
y-int$_{post}$	23.6649	0.9952	23.780	$p < .001^*$
y-int$_{pre}$ − y-int$_{post}$	5.0819	1.4074	3.611	$p < .001^*$
m$_{post}$	8.6582	1.0228	8.465	$p < .001^*$
m$_{pre}$ − m$_{post}$	0.3472	1.4464	0.240	$p = .810821$

a. Dependent Variable: Number of Wrong Answers.
   * Result was significant using $\alpha = .05$.

Results of the general linear model showed that there was not a significant difference in the slopes of the pretest and posttest simple linear regression models, m$_{pre}$ = 9.005 and m$_{post}$ = 8.658 (p = .810821, α = .05). However, there was a significant difference in their y-intercepts, y-int$_{pre}$ = 28.747 and y-int$_{post}$ = 23.665 ($p < .001$, α = .05). This significant, downward shift observed between the pretest and posttest models further supports the significant increase in overall mathematical content knowledge of prerequisite algebra skills that was detected earlier (see analysis for research question #1 on pp. 94-95).

Interesting Items

The five interesting items were grouped according to the sign of their residual value and the two groups were analyzed for any existing patterns. Again, only the general idea focused on by each item will be discussed. Recall that LMT regulations regarding non-released items prohibit discussing the specifics entailed in the questions themselves.

Outlying Residual. The only item whose residual was far enough from zero to be identified as an outlier in both models was question #29. This question involves using a part-to-part ratio and one partial quantity to calculate the total quantity (number concepts). To investigate specialized content knowledge, this item asks the person not to solve the problem, but to analyze two students' responses to the question and determine if either, neither, or both are on a correct solution path. Although both students' methods could eventually lead to a correct answer, 38 (out of the 48) participants erroneously answered that either one or both of the students were incorrect on the pretest, and 34 on the posttest.

Since this item had extremely large positive residuals in both models, it is apparent that this item was very difficult for students prior to taking the course and remained difficult even after course completion. It appears that upon entering this course, students did not have adequate understanding of the concepts in question #29, and that this course did not successfully address this particular gap in student knowledge.

Positive Residuals. The remaining three items (questions #6, #7, and #8a) with alarmingly high positive residual values ($e_7 = 13.22$ ($z_{e_7} = 1.94$), $e_6 = 11.58$ ($z_{e_6} = 1.70$), and $e_{8a} = 11.06$ ($z_{e_{8a}} = 1.63$)), represent the other questions missed by more students on the posttest than what was predicted by the posttest linear model. Note that all of these items also had positive residuals resulting from the pretest linear regression model, showing that these questions were also missed by more students on the pretest than what

was predicted by the pretest linear model. It appears that the content covered in these items were substantially difficult for students upon entering and exiting this course.

Although a slight improvement was observed for question #6 (34 incorrect answers on pretest, 32 incorrect answers on posttest), this item's residual increased from 8.63 (pretest) to 11.58 (posttest). This is due to the general shift that was observed between the pretest linear model and the posttest linear model (see Figure 24). Recall that there was a significant decrease in the y-intercept of the posttest linear model when compared to the pretest linear model. The improvement noted for question #6 was not large enough to compensate for this difference and resulted in this item having an increased residual value. A similar residual increase was noted for question #7, even though the exact same number of incorrect responses, 34, were given for this item on both the pretest and posttest. Again, due to the significant difference in the pretest and posttest linear models, this item's residual increased from 8.25 (pretest) to 13.22 (posttest). In contrast, a decrease was observed in the positive pretest and posttest residual values for question #8a, 18.63 and 11.06 respectively. For this item, however, there was an increase in the number of incorrect answers given on the posttest versus the pretest (25 incorrect on posttest, 20 incorrect on pretest). Even though the posttest residual was smaller, it appears that students actually did worse on this item on the posttest than they did on the pretest.

Of these three troublesome items, two are common content knowledge questions (questions #6 and #7) and one is a specialized content knowledge question (question #8a). All three questions, however, were categorized as addressing equation/function

concepts. These questions, on which students performed worse than predicted on both the pretest and posttest, deal with a variety of aspects of reading, writing, and representing formulas and functions and understanding equations being modeled within word problem contexts. Although this finding appeared to show some consistency among the type of concepts missed by more students than expected, it is interesting to note that this does not agree with the fact that the outlying positive residual for each administration resulted from a question which addresses number concepts.

Negative Residual. Only one of the items isolated from the posttest model had a dramatically small negative residual value, question #11b ($e_{11b} = -11.44$ ($z_{e_{11b}} = -1.68$)). Since this question was missed by less students on the posttest than predicted, students actually performed better than expected on this item. It should be noted, however, that this item also had a negative residual value, -11.31, on the pretest. Since students also performed better than expected on this item on the pretest, it appears that students already had substantial mastery of this content prior to taking this course. Question #11b addresses common content knowledge of equation/function concepts. This item provides a linear graph and asks about the relationship displayed between the two variables.

<div align="center">Summary of Important Results</div>

1. Upon completion of Math 130, a statistically significant increase was identified in preservice elementary teachers' total mathematical content knowledge of prerequisite algebra skills.

2. Upon completion of Math 130, a statistically significant increase was identified in preservice elementary teachers' common content knowledge of prerequisite algebra skills.

3. Upon completion of Math 130, a statistically significant increase was identified in preservice elementary teachers' specialized content knowledge of prerequisite algebra skills.

4. Upon completion of Math 130, a statistically significant increase was identified in preservice elementary teachers' content knowledge of number concepts.

5. Upon completion of Math 130, a statistically significant increase was identified in preservice elementary teachers' content knowledge of equation/function concepts.

6. A statistically significant positive correlation was found between preservice elementary teachers common content knowledge and specialized content knowledge of prerequisite algebra concepts ($r = .716$).

7. Using a linear model to predict the number of students to incorrectly answer each item based on each item's difficulty level, four items were identified as resulting in substantially worse student performance on the posttest than expected. The item with the most extreme, outlying residual asks preservice teachers to analyze students' work (specialized content knowledge) on ratios and proportions (number concepts). The remaining three items address a combination of common and specialized content knowledge of

equation/function concepts. These items test preservice elementary teachers' abilities to read, write, recognize, and represent a variety of formulas and functions. One item resulted in substantially better student performance on the posttest than predicted. This item addresses common content knowledge of the relationship between two variables displayed by a linear graph (equation/function concepts).

CHAPTER 5

CONCLUSIONS

Introduction

This chapter begins with an overview of the purpose and methodology of this study. Research results are first summarized in regards to the five research questions and then discussed more thoroughly with their implications and relation to current literature. Recommendations for practice are presented along with directions for future research. Lastly, conclusion for this study is provided.

Overview of the Study

More and more district and state high school graduation requirements are including algebra, increasing the need for all students, no longer just the college-bound, to be algebra proficient (Fey, 1989). Despite the obvious significance of algebra on a national scale, the National Assessment of Educational Progress shows a deficiency in the algebra achievement of U.S. students (Chazan & Yerushalmy, 2003). Research suggests that for students to succeed in Algebra I (or an equivalent first algebra course), it is vital they master prerequisite algebra concepts throughout their K-8 mathematics education: (1) numbers (and numerical operations), (2) ratios/proportions, (3) the order of operations, (4) equality, (5) patterning, (6) algebraic symbolism (including letter usage), (7) algebraic equations, (8) functions, and (9) graphing (Welder, 2006).

Research illustrates that student achievement is affected by teachers' knowledge, requiring elementary and middle school (K-8) teachers to have satisfactory knowledge of prerequisite algebra concepts (Fennema & Franke, 1992; Greenwald et al., 1996). This study's theoretical framework for the knowledge for teaching mathematics suggests that the mathematical content knowledge needed for teaching consists of specialized content knowledge in addition to common content knowledge (Ball, 2003, 2006; Hill & Ball, 2004; Hill et al., 2004; Rowan et al., 2001; Shulman, 1986). Specialized mathematical content knowledge extends beyond solving mathematical problems to encompass how and why mathematical procedures work and an awareness of structuring and representing mathematical content for learners.

This study investigated the effects of an undergraduate mathematics content course for preservice elementary teachers on their common and specialized content knowledge of prerequisite algebra concepts, using a pre-experimental one-group pretest-posttest design. A quantitative, 51-item, multiple-choice instrument, developed specifically to measure both types of content knowledge with respect to prerequisite algebra concepts, was conscientiously constructed from the Learning Mathematics for Teaching Project's Content Knowledge for Teaching Mathematics Measures question bank (LMT, 2006). This instrument was administered to all students enrolled in Math 130: Mathematics for Elementary Teachers I ($n = 48$), at Montana State University, during the first and last weeks of the fall semester of 2006.

Five matched pairs t-tests, comparing pretest and posttest scores within the single sample, were used to investigate the effects of Math 130 on preservice elementary

teachers' overall mathematical content knowledge, common content knowledge, and specialized content knowledge of prerequisite algebra concepts, as well as mathematical content knowledge of number concepts and equation/function concepts. Correlation analysis was applied to posttest common and specialized content knowledge scores to test the relationship between the common and specialized content knowledge of preservice teachers. Lastly, a one-parameter linear model was constructed to predict the number of participants to incorrectly answer each item on the instrument, based on item difficulty. Residuals were calculated and the five items whose residuals landed farthest from the least squares regression line were identified. These items, marked as being missed by notably more or less students than predicted by the linear model, were analyzed in terms of the content and type of knowledge they address.

Summary of Research Results

Research Questions 1-3: Effects on Prerequisite Algebra Knowledge

Upon completion of Math 130, statistically significant increases were identified in all five tested areas of preservice elementary teachers' knowledge of prerequisite algebra skills.

- Total mathematical content knowledge of prerequisite algebra concepts improved an average of .6430 pretest standard deviations; true mean improvement on the total instrument is believed to fall in the range of 7.8% to 12.14%.

- Mathematical content knowledge of number concepts improved an average of .7889 pretest standard deviations; true mean improvement in terms of number concepts is believed to fall in the range of 9.07% to 15.36%.

- Mathematical content knowledge of equation/function concepts improved an average of .3906 pretest standard deviations; true mean improvement in terms of equation/function concepts is believed to fall in the range of 4.04% to 10.09%.

- Common content knowledge of prerequisite algebra concepts improved an average of .5431 pretest standard deviations; true mean improvement in common content knowledge is believed to fall in the range of 6.0% to 11.74%.

- Specialized content knowledge of prerequisite algebra concepts improved an average of .6438 pretest standard deviations; true mean improvement in specialized content knowledge is believed to fall in the range of 7.15% to 16.18%.

Research Question 4: Relationship Between Common
and Specialized Content Knowledge

A Pearson's correlation coefficient of .716 showed a statistically significantly ($p < .001$, $\alpha = .05$) positive relationship between the preservice elementary teachers' common and specialized content knowledge of prerequisite algebra concepts.

<u>Research Question 5: Analyzing Interesting Items</u>

One item resulted in substantially better student performance on the posttest than predicted, whereas four items were identified as resulting in substantially worse student performance on the posttest than expected.

<u>Unchallenging Item (#11b).</u> Based on the difficulty level of question #11b, -0.604, the linear model constructed from this study's posttest data predicted 18.44 incorrect responses to be produced. However, only 7 of the 48 participants incorrectly answered this item or left it blank on the posttest. This question addresses common content knowledge of the relationship between two variables displayed by a linear graph (equation/function concept).

<u>Troublesome Item (Outlier #29)</u>. This item, the only true outlier with posttest residual $e_{29} = 19.54$, asks preservice teachers to analyze students' work (specialized content knowledge) on ratios and proportions (number concepts). The posttest linear model predicted 14.46 incorrect answers, based on the item's difficulty level of -1.063. Alarmingly, 34 of the 48 participants either did not answer or incorrectly answered this item on the posttest.

Question #29 shows the works of two students who were asked to use a part-to-part ratio and one partial quantity to solve for the total quantity. Each student approaches the problem in a different way and the beginning steps of each one's work is shown. The item then asks the preservice teacher to evaluate if either, neither, or both students could potentially use their chosen approach to correctly solve the problem. Although both

solution paths could be used to successfully answer the posed question, 21 participants believed that only one of the two approaches was valid, nine thought neither approach would work, and three did not provide an answer to this item.

Troublesome Items (Non-Outliers #6, #7, and #8a). Difficulty levels of -0.375, -0.3328, and -1.1232, for questions #6, #7, and #8a respectively, created predictions of 20.42, 20.78, and 13.94 incorrect responses on the posttest. However, the number of participants that did not respond or incorrectly responded to questions #6, #7, and #8a on the posttest were 32, 34, and 25, respectively. These three items address a combination of common and specialized content knowledge of equation/function concepts. These items test preservice elementary teachers' abilities to read, write, recognize, and represent a variety of functions and formulas.

Discussion of Results

Number Concepts

Mastery of number concepts and numerical operations are fundamental to a student's ability to learn algebra. Booth (1984) claims that elementary algebra students' difficulties are caused by confusion surrounding computational ideas, including inverse operations, associativity, commutativity, distributivity, and the order of operations convention. These misconstrued ideas are among basic number rules essential for algebraic manipulation and equation solving (Watson, 1990).

To address the importance of number concepts, the instrument constructed for this study assessed knowledge of several types and forms of numbers and numerical

operations. In fact, items specifically addressed whole number operations, subtraction of integers, representations and explanations of fractions and fraction operations, decimal representations, prime numbers, and the order of operations.

This study showed a significant growth in preservice teachers' mathematical content knowledge of number concepts. This finding is consistent with the design of the Math 130 course curriculum, which is constructed to focus on numbers and numerical operations. Therefore, it is not surprising that the largest increase in knowledge was in the field of numerical concepts. Math 130 does appear to improve preservice teachers' knowledge of the content it purports to teach.

Ratios and Proportions

Although it is clear that students achieved a significant growth in overall number knowledge over the course of the semester, it is important to note that the only item with a large enough residual value to be identified as an outlier (question #29) deals with ratios and proportions, a number concept. In fact, 71% of the participants incorrectly answered this item on the posttest, for which the linear model predicted only 29% of answers to be wrong. Only two additional items on the instrument (questions #9 and #16) addressed similar ideas. Thirty-three incorrect responses (out of 48) were given for question #16 on the posttest, when only 24.27 were predicted, resulting in this item having the seventh largest positive residual ($e_{16} = 8.73$). Furthermore, although the residual value for question #9 was nearly zero ($e_9 = 0.31$), 41 of the 48 preservice teachers answered this item incorrectly on the posttest($\hat{y}_{wrong_9} = 40.69$). Performance on

the three items addressing ratios and proportions (questions #9, #16, and #29) collectively suggest that preservice teachers' knowledge is lacking in this field.

The Math 130 curriculum, including both text material and in-class activities, indicates that ratios and proportion ideas were thoroughly discussed during two class days. The students cooperatively completed an extensive activity on "Exploring Proportion," (Lappen et al., 1998, pp. 7, 27-28, 32, 44-45, 53-54, 57-58) which helped them practice their abilities to interpret and compare ratios, estimate large populations, and work with population density. Furthermore, this material was covered early in November 2006, only four weeks prior to the posttest administration of this study's instrument. Although students were exposed to ratios and proportions throughout the Math 130 curriculum, it appears that they did not achieve full mastery of this content or at least were not able to apply their knowledge to the questions on this instrument.

This finding is not surprising, however. Not only have ratios and proportions been found to be notoriously difficult for children and adolescents (Post et al., 1988), this content remains troublesome for many college students (Ilany, Keret, & Ben-Chaim, 2004). In fact, "there is evidence that a large segment of our society never acquires fluency in proportional thinking" (Hofer, 1988, p. 285). The results of this study support the numerous research findings that have indicated "many gaps in the content knowledge of preservice and inservice teachers in mathematical subjects taught in elementary and middle schools, including the topics of ratio and proportion" (Ilany et al., 2004, p. 81). Perhaps this misunderstanding can be attributed to the fact that proportional reasoning requires a solid understanding of several rational number concepts including order and

equivalence, the relationship between a unit and its parts, the meaning and interpretation of ratio, and various division issues (Post et al., 1988). The area of integer division, and hence fractions, is known to be a source of student errors (Kieran, 1988). In fact, Wu (2001) believes that K-12 teachers are not currently teaching fractions at a deep enough level to prepare students for algebra.

Misconceptions in the area of ratio and proportion are particularly noteworthy due to their importance in current mathematics curricula (Lo & Watanabe, 1995). In fact, proportionality is considered to be an important contributor to students' development of pre-algebraic understanding. Because proportions can provide wonderful examples of naturally occurring linear functions, Post et al. (1988) feel that proportionality has the ability to connect common numerical experiences and patterns, with which students are familiar, to more abstract relationships in algebra. Proportions can also be used to introduce students to algebraic representation and variable manipulation in a way that parallels their knowledge of arithmetic.

Due to the significance of ratios and proportions, which are commonly misunderstood, one might consider increasing the number of days dedicated to the study of this topic in the Math 130 curriculum. Another suggestion is to have students repeatedly use ratio and proportion ideas by embedding them within subsequent course material, instead of strictly designating two days for the study of this content. Regardless of how the curriculum is structured, the work of Ilany et al. (2004) demands that the teaching of ratio and proportion topics to preservice teachers be given careful consideration. Ilany et al. developed a model for teaching ratio and proportion topics in

mathematics teacher education courses. Their model incorporates "authentic investigative activities with five types of activities: Introductive activities, investigative activities dealing with ratio, dealing with rate, dealing with scaling, and dealing with indirect proportion" (Ilany et al., 2004, p. 82). These activities, which involve small and large integer numbers, fractions, decimals, and percents, include making quantitative and qualitative numerical comparisons between ratios and finding a missing value. Ilany et al. have successfully shown that their model has led to preservice teachers acquiring both mathematical content knowledge and pedagogical-didactical knowledge. Therefore, the selection and implementation of course activities may be an important piece of addressing preservice teachers' misunderstandings about ratios and proportions.

It is worth noting that each of the two class days dedicated to the study of ratios and proportions were followed by a day on which the university did not hold classes. Ratios and proportion content was covered on Monday, November 6, and Thursday, November 9, which were immediately followed by Election Day and Veteran's Day, respectively. It is unclear whether the disconnected way this material was presented had any effect on students' ability to answer the proportion items on the administered instrument (particularly question #29). With the week's interrupted course schedule, it is also possible that some students were absent for a portion of or the entire week and consequently missed both days dedicated to this topic. However, due to the results of this study and the fact that ratios and proportions are notoriously misunderstood (Singh, 2000), it could be valuable to be aware of where this content falls in the calendar year;

course curriculum could be rearranged to minimize the interference of vacation days and attain optimal emphasis and continuity.

Equation/Function Concepts

The functional relation between two variables is a central concept in prealgebra courses. According to Brenner et al. (1995), translating and applying mathematical representations of functional relations are two cognitive skills that are essential for success in algebraic reasoning. It is therefore encouraging to see significant growth in preservice teachers' knowledge of equation/function concepts after their completion of Math 130. In addition, the one question on the administered instrument identified as receiving fewer incorrect responses than predicted is in the area of equations/functions. Students performed exceptionally well on the item that addresses the relationship between two variables displayed by a linear graph. This result agrees with previous findings showing that students handle functions better when they are given in graphical versus algebraic form (Markovits et al., 1988).

These results, however, could be considered somewhat surprising, given that the Math 130 curriculum clearly focuses on numbers and numerical operations rather than on equations and functions. In fact, only three class days were dedicated to the study of relations and functions. The students spent two class days discussing this content and a third class day cooperatively completing an activity, "Fun with Functions!" (Willard, 2006). This activity asked students to represent functional relationships, described in words, in multiple ways using symbols, tables, and graphs. Furthermore, this brief introduction to functions and relations was presented at the end of September 2006,

almost 11 weeks prior to the instrument's posttest administration. The significant growth in equation/function knowledge could be attributed to students being reacquainted with material they have encountered in the past but have since forgotten. For many students, Math 130 is the first mathematics course they have taken since their sophomore or junior year of high school. Perhaps it is merely the review of mathematical content that is leading to observed growth in the field of equations and functions.

Despite the significant growth observed after seemingly minimal coverage of functions and relations, there is clearly room for improvement in this area. Recall that, three of the four items identified as troublesome address equation/function concepts. In part, these three items ask students to: (1) write a formula to represent a geometric pattern, (2) identify which of multiple representations of functional relationships (including a geometric pattern, a story-problem, and algebraic statements) correspond to the same relationship, and (3) interpret functional relationships described within written story-problem contexts.

The consistency found among these three items is not surprising considering that functions are notoriously challenging (Brenner et al., 1995). Not only do letter usage and algebraic notation cause difficulties for many students (Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Sleeman, 1984), symbol confusion, misconceptions of equality, the conventional method of reading conversions (Booth, 1986), and reversal errors (Clement et al., 1981; Wollman, 1983) cause students to incorrectly read algebraic equations and translate word sentences into algebraic equations. Therefore, it could also

be beneficial to increase the numbers of days dedicated to the study of this topic in the Math 130 curriculum.

Common Content Knowledge

The content covered throughout the Math 130 curriculum does not exceed that which is typically covered in K-8 classrooms. Furthermore, common content knowledge only requires that a person possess the skills and procedures necessary for solving (not explaining or representing) mathematical problems. The common content knowledge tested by the instrument designed for this study only investigated students' abilities to solve K-8 level mathematics problems dealing with prerequisite algebra concepts, and yet significant growth was noted upon the completion of Math 130. This finding supports existing literature that establishes preservice and inservice elementary teachers' content knowledge as insufficient and/or identifies gaps in teachers' content knowledge (Ball, 1990, Ball & Wilson, 1990, Ilany et al., 2004). Therefore, the need for courses such as this one, heavy in content and dedicated to K-8 mathematical content for future elementary teachers, is reinforced.

Mastery of this level of mathematical computation was assumed due to the prerequisites for Math 130 enrollment. In order to enroll in this course, students have to meet one of the following four requirements:

- Successful completion of Introductory Algebra (or a higher level mathematics course) with a grade of D or better
- Successful completion of Montana State University's Math Placement Test at Level III (which allows enrollment in College Algebra)

- ACT math score of 23 or higher

- SAT math score of 530 or higher

One would expect these prerequisite options to demand a substantial mastery of K-8 computational mathematics. However, for the semester under investigation, course prerequisites were not strictly enforced. This may have influenced the level of prerequisite knowledge students brought with them to the course and hence influenced the growth that was identified in common content knowledge of K-8 level mathematics.

Specialized Content Knowledge

In the field of mathematics, how teachers hold knowledge may matter more than how much knowledge they hold (Hill & Ball, 2004). In fact, "teaching quality might not relate so much to performance on standard tests of mathematics achievement as it does to whether teachers' knowledge is procedural or conceptual, whether it is connected to big ideas or isolated into small bits, or whether it is compressed or conceptually unpacked" (Hill & Ball, 2004, p. 332). Researchers assert that this additional knowledge required of teachers (or lack thereof) will affect their teaching decisions and ultimately their students' achievements in mathematics (Ball & Wilson, 1990; Graeber, 1999; Lee et al., 2003; Rine, 1998).

Practical experience as a teacher was once believed by scholars to be the best way for a person to acquire any aspect of pedagogical content knowledge. In fact, collegiate teacher education was thought to be incapable of making significant contributions to what teachers need to know or be able to do (Ball & Wilson, 1990). However, the potential power of content courses to enhance such knowledge has been demonstrated by the work

of Davis and McGowen (2001), who illustrated that the mathematical understanding of a preservice elementary teacher significantly improved during a mathematics content course. The current study extends Davis and McGowen's valuable research beyond a singular case study to show that similar results can be found for larger groups of students.

Since Math 130 is one of only two required mathematics content courses for elementary education majors at Montana State University, research recommends that this course be used to develop specialized content knowledge, the content knowledge that is specific to the needs of teachers (Stacey et al., 2001). Battista (1994) urges teacher education institutions to offer numerous mathematics courses for teachers that treat mathematics as sense making, rather than rule following. Teachers must be taught mathematics properly before they can be expected to teach it properly. Furthermore, Battista (1994) warns that simply having preservice teachers take more college-level mathematics courses will not adequately prepare them to teach elementary mathematics. Most university mathematics courses merely reinforce the view of mathematics as a set of memorized procedures; hence, taking more of them will not benefit preservice elementary teachers in the area of specialized content knowledge (Battista, 1994).

Alarmingly, "few mathematics courses offer opportunities to learn mathematics in ways that would produce such knowledge" (Ball, 2003, p. 8). University courses required of preservice elementary teachers often do not have the time or concentration needed to develop the mathematical knowledge that is essential for elementary teachers (Battista, 1994). Counter to previous findings (see Ball & Wilson, 1990), the results of the current study show a significant increase in preservice elementary teachers' specialized content

knowledge of prerequisite algebra concepts upon completion of Math 130. Therefore, this study provides additional evidence in support of the numerous scholars who argue that it is not only necessary, but in fact possible, for teacher educators to advance preservice teachers' pedagogical content knowledge within collegiate course settings (Battista, 1994; Chen & Ennis, 1995; Davis & McGowen, 2001; Manouchehri, 1996; Miller, 1999; Stacey et al., 2001).

This finding also supports the structure and delivery of this content course for elementary education majors. The Math 130 course curriculum, which is considered to be standard for this type of mathematics content course offered for elementary education majors, sequentially followed Chapters 1-9 of a traditional textbook, *Mathematics for Elementary Teachers: A Contemporary Approach, 7th edition* (Musser et al., 2006), with only minor deviations and supplementary materials (see Chapter 3, pp. 63-67 for complete course description).

The teachers of Math 130 incorporated multiple strategies through a variety of materials, manipulatives, and hands-on activities. The preservice teachers were also exposed to students' thinking and common errors through examples of K-8 students' work. It can be very time-consuming to integrate aspects of mathematics teaching that are typically considered to be methodology into content courses. However, the significant growth in specialized content knowledge observed through this study encourages the continuation of practices, such as these, that may help students extend their knowledge beyond that which is considered common content knowledge. Furthermore, this study showed a strong, positive relationship between preservice teachers' specialized and

common content knowledge of prerequisite algebra skills. The fact that both types of knowledge can be enhanced simultaneously provides further incentive to blend the ideas and practices that are generally divided between content and methodology courses.

It is noteworthy that these findings are counter to those of Hill and Ball (2004), who suggest that specialized knowledge for teaching mathematics may exist independently from common mathematical knowledge. Additionally, Ball et al. (2005) found that the results from questions asked of inservice teachers regarding specialized knowledge of mathematics were statistically separable from those found from common content knowledge items.

<p style="text-align:center;">Recommendations for Practice</p>

If elementary teachers are to successfully prepare students to learn algebra, they need to be properly prepared to teach fundamental concepts surrounding the ideas of equations and functions. The results of this study suggest that collegiate content courses for elementary education majors should dedicate substantial time to the study of equation/function concepts, especially the reading, writing, and representing of functions and formulas. Math 130 is an essential piece of preservice elementary teachers' preparation to teach algebra, and this study suggests that dedicating even as few as three class days to the study of functions and relations significantly helped advance preservice teachers' knowledge of these concepts. Therefore, it is vital that teacher educators continue to address this content in content courses for elementary education students. The fact that three of the four items identified as being troublesome for preservice teachers

addressed equation/function ideas further supports increasing student exposure to equation/function content.

The three troublesome items pertaining to equation/function ideas addressed: (1) analyzing multiple representations of functional relationships, (2) interpreting functional relationships described within written story-problems, and (3) writing formulas. This finding calls attention to the need, stated by Manouchehri (1996), for development of representations and representational contexts that will enable teachers to draw connections between concepts and applications. In addition, students must be taught to ask themselves questions regarding the equations that they write and to create meaningful ways of checking their answers. Wollman (1983) states that with tools such as these, teachers could help strengthen students' fluency in writing equations.

Collegiate content courses for elementary education majors should also work to enhance student understanding and correct student misconceptions regarding ratios and proportions. "Proportional reasoning is at the heart of mathematics in the upper grades of the elementary schools and in the middle schools…Therefore, the topics of ratio and proportion should have central part in mathematics curriculum for children in school as well as for preservice mathematics teacher education" (Ilany et al., 2004, p. 81). Suggestions include increasing the number of days dedicated to this topic beyond the current two, repeatedly using the content by embedding ratio and proportion ideas within subsequent course content, or integrating a research-based model of teaching ratios and proportions, such as the one suggested by Ilany et al. (2004). Furthermore, curriculum design should give special consideration to notoriously difficult content such as ratio and

proportions. In order to maximize student learning, content that has been proven to be troublesome needs to be carefully placed into course schedules to attain optimal emphasis and continuity.

It is also important for content courses to expose students to multiple strategies, through a variety of materials, manipulatives, and hands-on activities, such as those used in the Math 130 course under investigation. Although it can be very time-consuming to integrate aspects of mathematics teaching that are typically considered to be methodology into content courses, the results of this study encourages the continuation of these practices. The strong, positive relationship between preservice teachers' specialized and common content knowledge and the significant growth of specialized content knowledge identified through this study provide incentive to blend content and pedagogy ideas and practices together in a way that can enhance both types of knowledge simultaneously.

Lastly, due to the significant increase found in the common content knowledge of Math 130 students, it is clear that these preservice teachers had not mastered computational skills of K-8 level prerequisite algebra concepts prior to taking this class. Therefore, the implementation and enforcement of prerequisites for content courses for preservice teachers, such as this one, is recommended.

Directions for Future Research

Effective means of defining and testing the knowledge of preservice teachers are necessary to assure successful teacher preparation. Levels of adequacy in terms of both common and specialized content knowledge need to be established so that preservice

teachers can be accurately evaluated upon the completion of their collegiate studies, prior to certification. Since researchers have asserted that various types of knowledge are essential for effective teaching (Ball, 2003, 2006; Hill & Ball, 2004; Hill et al., 2004; Rowan et al., 2001; Shulman, 1986), certification exams should be carefully examined to assure that items are addressing more than teachers' common content knowledge alone.

Not all teacher preparation programs offer a content course which directly addresses K-8 mathematics. Some programs require elementary education students to take college-level or general education mathematics courses, such as College Algebra or Calculus. However, Battista (1994) claims that additional college-level mathematics courses are not capable of enhancing specialized content knowledge. It would be very valuable to test Battista's assertion by comparing groups of preservice elementary teachers during and after their completion of programs with differing mathematics course requirements.

Research has shown that methods courses (Ward et al., 2003) and practical experiences as a teacher (Ball & Wilson, 1990) have successfully increased various aspects of teachers' pedagogical content knowledge. Therefore, this study should be extended into a longitudinal study that not only tests preservice teachers' knowledge before and after completion of content courses, but again after completion of methods courses and student teaching. Only then will teacher educators truly discover where and how mathematics teachers' content knowledge, particularly that which is specialized, is developed and be able to encourage its growth.

Testing teachers' mathematical knowledge will not benefit K-8 students, however, if teacher knowledge does not translate into the classroom. The researcher's future work will broaden this exploration to investigate the ways in which teacher knowledge is utilized within the mathematics classroom. How teachers can and do use their mathematical content knowledge when teaching will be examined. Specifically, how does a teachers' specialized content knowledge affect the activities done in the classroom and/or the opportunities provided students?

The ultimate goal of this research is to increase student achievement; therefore this work should be extended to enhance the understanding of how teacher knowledge affects student learning. Research has already suggested that teachers' knowledge does have an effect on student achievement (Hill et al., 2005); however, more studies are needed to confirm this finding.

Lastly, since algebra preparation needs to occur within the K-8 mathematics curriculum, this research has focused on elementary and middle school teachers. The design and methodology of this study, however, could easily be extended to apply to other grade bands or content areas.

Conclusion

Research suggests that for students to succeed in Algebra I (or an equivalent first algebra course), they must master both number and equation/function concepts (see p. 2 for specific algebra prerequisites and pp. 71-72 for concept definitions) throughout their K-8 mathematics education (Welder, 2006). Furthermore, research has illustrated

that student achievement is affected by teachers' knowledge (Fennema & Franke, 1992; Greenwald et al., 1996). In fact, what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn (Fennema & Franke, 1992). Hence, for students to be properly prepared to learn algebra, elementary and middle school (K-8) teachers must have satisfactory knowledge of prerequisite algebra concepts.

The mathematical content knowledge needed for effective teaching consists of pedagogical content knowledge, in addition to common content knowledge (Ball, 2003, 2006; Hill & Ball, 2004; Hill et al., 2004; Rowan et al., 2001; Shulman, 1986) (see pp. 4-5 for knowledge framework and definitions). Since the enhancement of pedagogical content knowledge can help connect one's subject content knowledge with the curriculum delivered in classrooms (Chen & Ennis, 1995), it is vital that K-8 teachers have the opportunity to develop this knowledge in terms of prerequisite algebra concepts.

The current study found that preservice elementary teachers' knowledge successfully grew in both common and specialized content knowledge in the areas of numbers, equations, and functions over the course of a one-semester, undergraduate content course designed for elementary education students. These findings validate the usefulness of content specific courses for teachers, such as this one, and the ability of collegiate courses to enhance understanding beyond what is considered common content knowledge. Specific content areas, namely ratios/proportions and the writing and representing of functions and formulas, were identified as troublesome for preservice teachers, and recommendations for teacher educators were made.

Increasing teachers' mathematical content knowledge of prerequisite algebra concepts will positively affect the algebra achievement of future K-8 students. To that end, the work of this study offers results and recommendations to advance the content knowledge of preservice and inservice mathematics teachers by guiding and informing collegiate teacher preparation courses.

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